THE SPECTRAL SEMIDISTANCE IN BANACH ALGEBRAS

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Let A be a Banach complex algebra with identity $\mathbf{1}$. For elements $a, b \in A$ the spectral semidistance between a and b is defined as follows: Denote the commutator $C_{a,b} := L_a - R_b$, and then consider powers evaluated at $\mathbf{1}$:

$$C_{a,b}^n \mathbf{1} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} a^{n-k} b^k.$$

Writing

$$\rho(a,b) := \limsup_n \left\| C_{a,b}^n \mathbf{1} \right\|^{1/n}$$

we define the spectral semidistance

$$d(a,b) := \sup\{\rho(a,b), \rho(b,a)\}.$$

The spectral semidistance is a semimetric and could be viewed as the noncommutative generalization of the distance induced by the spectral radius in the commutative case. If $d(a,b) = 0$ then a and b are called quasinilpotent equivalent.

The spectral semidistance between decomposable operators $S, T \in \mathcal{L}(X)$ can be formulated in terms of spectra via Vasilescu's geometric formula:

$$d(S, T) = \sup\{\Delta(\sigma_T(x), \sigma_S(x)) : 0 \neq x \in X\}$$

where $\sigma_S(x)$ and $\sigma_T(x)$ are, respectively, the local spectra of S and T at $x \in X$.

THEOREM

Suppose $\sigma(a)$ and $\sigma(b)$ are finite with $\sigma(a) = \{\lambda_1, \ldots, \lambda_n\}$, $\sigma(b) = \{\beta_1, \ldots, \beta_k\}$. If $\{p_1, \ldots, p_n\}$ and $\{q_1, \ldots, q_k\}$ are the corresponding Riesz projections then

$$\rho(a, b) = \sup\{|\lambda_i - \beta_j| : p_i q_j \neq 0\}. \tag{1}$$

THEOREM

Suppose $\sigma'(a)$ and $\sigma'(b)$ are discrete sets which cluster at $0 \in \mathbb{C}$, if anywhere. If $\sigma'(a) = \{\lambda_1, \lambda_2, \ldots\}$ and $\sigma'(b) = \{\beta_1, \beta_2, \ldots\}$ denote the nonzero spectral points of a and b, and if $\{p_1, p_2, \ldots\}$ and $\{q_1, q_2 \ldots\}$ are the corresponding Riesz projections, then ϱ takes at least one of the following values:

(i) $\rho(a, b) = \sup\{|\lambda_i - \beta_j| : p_i q_j \neq 0\}$, *or*

(ii) $\rho(a, b) = |\lambda_i|$ *for some* $i \in \mathbb{N}$, *or*

(iii) $\rho(a, b) = |\beta_i|$ *for some* $i \in \mathbb{N}$.

Moreover, $\rho(a, b) = 0$ if and only if the spectra and the corresponding Riesz projections of a and b coincide.

Let f be an entire function from $\mathbb{C}$ into a Banach algebra A. Then f has an everywhere convergent power series expansion

$$f(\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n,$$

with coefficients a_n belonging to A. Define a function

$$M_f(r) = \sup_{|\lambda| \leq r} \|f(\lambda)\|, \quad r > 0.$$

DEFINITION

The function f is said to be of finite order if there exists $K > 0$ and $R > 0$ such that $M_f(r) < e^{r^K}$ holds for all $r > R$. The infimum of the set of positive real numbers, K, such that the preceding inequality holds is called the order of f, denoted by ω_f. If $\omega_f = 1$ then f is said to be of exponential order.

DEFINITION

Suppose f is entire, and of finite order $\omega := \omega_f$. Then f said to be of finite type if there exists $L > 0$ and $R > 0$ such that $M_f(r) < e^{Lr^{\omega}}$ holds for all $r > R$. The infimum of the set of positive real numbers, L, such that the preceding inequality holds is called the type of f, denoted by τ_f.

It is well-known that the order and type of A-valued entire functions are given by the formulae

$$\omega_f = \limsup_n \left(\frac{n \log n}{\log \|a_n\|^{-1}} \right) \text{ and } \tau_f = \frac{1}{e\omega_f} \limsup_n \left(n \sqrt[n]{\|a_n\|^{\omega_f}} \right).$$

Let $a, b \in A$, and define

$$f : \lambda \mapsto e^{\lambda a} e^{-\lambda b}, \qquad \lambda \in \mathbb{C}.$$

The corresponding series expansion, valid for all $\lambda \in \mathbb{C}$, is given by

$$f(\lambda) = e^{\lambda a} e^{-\lambda b} = \sum_{n=0}^{\infty} \frac{\lambda^n C_{a,b}^n \mathbf{1}}{n!}.$$

The important observations for us are the following:

- f is of order at most one
- If f is of order precisely one, then the type of f is given by $\rho(a, b)$.

THEOREM

Suppose $d(a,b)=0$, and suppose f is analytic on an open set U containing $\sigma(a)=\sigma(b)$. Then $d(f(a),f(b))=0$.

THEOREM (Pólya)

Let $f:\mathbb{C}\to A$ be an entire function from the field $\mathbb{C}$ into a Banach algebra A. If f is (norm) bounded over $\mathbb{Z}$ and if

$$\limsup_{r\to\infty}\frac{\log M_f(r)}{r}\leq 0,$$

then f is constant.

THEOREM

Let A be a Banach algebra, and let $a,b\in A$. If $0\notin\sigma^*(a)$, then $a=b$ if and only if a and b are quasinilpotent equivalent, and $\{\|a^nb^{-n}\|:n\in\mathbb{Z}\}$ is bounded. More generally, two elements, a and b, in a Banach algebra coincide if and only if they are are quasinilpotent equivalent, and there exists $\alpha\notin\sigma^*(a)$ such that $\{\|(\alpha+a)^n(\alpha+b)^{-n}\|:n\in\mathbb{Z}\}$ is bounded.

COROLLARY

Let A be a C^*-algebra, and let $a, b \in A$ be both self-adjoint or both be unitary . Then $a = b$ if and only if a and b are quasinilpotent equivalent.

THEOREM (Brits & Raubenheimer)

Let A be a C-algebra. If a and b are normal elements of A, and if 0 is the only possible accumulation point of $\sigma(a)$, then $d(a, b) = 0$ if and only if $a = b$.*

The above results can be improved to hold for arbitrary normal elements using the fact that the spectral semidistance is related to the growth characteristics of an entire function from $\mathbb{C}$ to A. For this we shall need a typical version of the Phragmén-Lindelöf device.

THEOREM (Phragmén-Lindelöf)

Let u be a subharmonic function on the half-plane $H := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$, such that for some constants $A, B < \infty$

$$u(\lambda) \le A + B|\lambda|, \quad \lambda \in H. \tag{2}$$

If

$$\limsup_{\lambda \to \zeta} u(\lambda) \le 0 \text{ for all } \zeta \in \partial H \setminus \{\infty\} \tag{3}$$

and if

$$\limsup_{t \to \infty} \frac{u(t)}{t} = L, \quad (t \in \mathbb{R}^+) \tag{4}$$

then

$$u(\lambda) \le L \operatorname{Re}\lambda, \quad \lambda \in H. \tag{5}$$

THEOREM

Let A be a C^*-algebra and let $a, b \in A$ be normal elements. Then $a = b$ if and only if a and b are quasinilpotent equivalent.

SKETCH OF PROOF

- *Define entire functions, f and g, from $\mathbb{C}$ into A by respectively*

  $$f(\lambda) = e^{\lambda ia} e^{-\lambda i(b-b^*)} e^{-\lambda ia^*} \text{ and } g(\lambda) = e^{\lambda i(a-a^*)} e^{-\lambda i(b-b^*)},$$

  and notice that $r_\sigma(f(\lambda)) = r_\sigma(g(\lambda))$ for all $\lambda \in \mathbb{C}$.
- *Then define*

  $$\mathbb{C} : \lambda \mapsto \log r_\sigma(f(\lambda))$$

  which is subharmonic on $\mathbb{C}$.
- *Applying the growth characteristics it follows that, given $\epsilon > 0$ arbitrary, there exists $R(\epsilon) > 0$ such that for all $r > R(\epsilon)$*

  $$\log r_\sigma(f(\lambda)) \leq \log \left(\|e^{\lambda ia} e^{-\lambda ib}\| \, \|e^{\lambda ib^*} e^{-\lambda ia^*}\| \right) \leq \epsilon r$$

  whenever $|\lambda| \leq r$, hence (2) holds.

- To establish (3): if ζ lies on the imaginary axis, then

$$\limsup_{\lambda \to \zeta} \log r_\sigma(f(\lambda)) = \limsup_{\lambda \to \zeta} \log r_\sigma(g(\lambda)) \leq 0$$

- Finally, one also obtains

$$\limsup_{\substack{t \to \infty \\ t > 0}} \frac{\log r_\sigma(f(t))}{t} = 0,$$

whence it follows that $r_\sigma(f(\lambda)) = r_\sigma(g(\lambda)) \leq 1$ for all $\lambda \in \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda \geq 0\}$.

- Using a symmetric argument we can prove the same result for $-H$, and hence that $r_\sigma(f(\lambda)) = r_\sigma(g(\lambda)) \leq 1$ for all $\lambda \in \mathbb{C}$.
- Now define an entire function

$$h(\lambda) = \begin{cases} \left(e^{\lambda i(a-a^*)} e^{-\lambda i(b-b^*)} - 1\right) / \lambda & \text{if } \lambda \neq 0 \\ i(a-a^*) - i(b-b^*) & \text{if } \lambda = 0. \end{cases}$$

- Since $r_\sigma(g(\lambda))$ is bounded on $\mathbb{C}$ it follows that $\limsup_{|\lambda|\to\infty} r_\sigma(h(\lambda)) = 0$. But $r_\sigma(h(\lambda))$ is subharmonic on $\mathbb{C}$ and therefore, by Liouville's Theorem (for subharmonic functions), it must be constantly zero on $\mathbb{C}$
- In particular, we see that $r_\sigma(i(a-a^*) - i(b-b^*)) = 0$. But $i(a-a^*) - i(b-b^*)$ being self-adjoint it follows that $a - a^* = b - b^*$.
- Writing $c := a - a^* = b - b^*$ we see that c commutes with both a and b from which we then obtain

$$d\left(\frac{a+a^*}{2}, \frac{b+b^*}{2}\right) = d\left(a - \frac{c}{2}, b - \frac{c}{2}\right) = d(a,b) = 0.$$

So, using a preceding corollary, we get $a + a^* = b + b^*$ and hence that $a = b$ as required.