Derivations from the disc algebra into natural modules

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Setting the scene

$\mathbf{D}$ = the open unit disc in $\mathbf{C}$; $\mathbf{T} = \partial\mathbf{D}$ the unit circle.

$\mathcal{O}(\mathbf{D})$ = the algebra of holomorphic functions $\mathbf{D} \to \mathbf{C}$.

The disc algebra

$$\begin{aligned} \mathrm{A}(\mathbf{D}) &= \{f \in \mathrm{C}(\overline{\mathbf{D}}) \text{ and } f|_{\mathbf{D}} \in \mathcal{O}(\mathbf{D})\} \\ &\cong \{f \in \mathrm{C}(\mathbf{T}) : \widehat{f}(n) = 0 \text{ for all } n < 0\} \end{aligned}$$

Fundamental example in Banach algebra theory.

We know many things about $\mathrm{A}(\mathbf{D})$, but not so much about its continuous Hochschild cohomology. This talk explores a small corner.

WHY CARE ABOUT $A(\mathbf{D})$?

- Completion in a natural norm of the polynomial ring $\mathbf{C}[z]$, which has a basic role in commutative algebra/algebraic geometry.
- Has connections with operator theory (e.g. von Neumann's inequality).
- Serves as a prototype/motivation for more exotic Banach algebras of functions with analytic structure.

Indirect motivation: the paper

M. C. WHITE, *Injective modules over uniform algebras*.
Proc. LMS **73** (1996) 155–184.

DERIVATIONS

Let A be an algebra and M an A-bimodule. A derivation is a linear map $D : A \to M$ satisfying

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad \text{for all } a, b \in A.$$

(In this talk, all derivations etc are tacitly **norm-continuous**.)

$\mathsf{Der}(A, M)$ = space of derivations $A \to M$.

In this talk, only consider **symmetric** bimodules ($a \cdot m = m \cdot a$).

If $D \in \mathsf{Der}(\mathrm{A}(\mathbf{D}), M)$ an easy induction gives

$$D(Z^n) = nZ^{n-1} \cdot D(Z) \quad (n \geq 1)$$

so by linearity

$$D(f) = f' \cdot D(Z) \text{ for every } f \in \mathbf{C}[z].$$

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Easy result

Let M be a $\mathrm{C}(\mathbf{T})$-bimodule and $D : \mathrm{A}(\mathbf{D}) \to M$ a derivation.
Then $D = 0$.

Hence: $\mathsf{Der}(\mathrm{A}(\mathbf{D}), \mathrm{C}(\mathbf{T})) = 0$; and $\mathsf{Der}(\mathrm{A}(\mathbf{D}), \mathrm{A}(\mathbf{D})) = 0$.

DERIVATIONS INTO $C(\mathbf{T})/A(\mathbf{D})$

Notation

Define $\mathsf{P}_+ : L^1(\mathbf{T}) \to \mathcal{O}(\mathbf{D})$ and $\mathsf{P}_- : L^1(\mathbf{T}) \to \operatorname{Conj}\mathcal{O}(\mathbf{D})$ by

$$(\mathsf{P}_+ f)(z) = \sum_{n\geq 1} \widehat{f}(n) z^n \quad , \quad (\mathsf{P}_- f)(z) = \sum_{n\geq 1} \widehat{f}(-n) \overline{z}^n$$

For $k \in L^1(\mathbf{T})$ let ∂k be the formal/distributional derivative of k:

$$\widehat{\partial k}(n) = in\,\widehat{k}(n) \qquad (n \in \mathbf{Z})$$

H^p denotes Hardy space on the disc/circle ($1 \leq p \leq \infty$).

Explicit description of $\mathsf{Der}(\mathrm{A}(\mathbf{D}), \mathrm{C}(\mathbf{T})/\mathrm{A}(\mathbf{D}))$

The following theorem paraphrases results of
Aleksandrov–Peller, IMRN 1996.

Theorem

Let $k \in \mathrm{C}(\mathbf{T})$. Then the following are equivalent:

1. $\|f'k + \mathrm{A}(\mathbf{D})\|_{\mathrm{C}(\mathbf{T})/\mathrm{A}(\mathbf{D})} \lesssim \|f\|_\infty$ *for all* $f \in \mathbf{C}[z]$*;*
2. $\partial k \in (H^1)^*$, *i.e.*

$$\sup \left\{ \left| \int_{\mathbf{T}} \partial k(z) h(z)\, |dz| \right| \;:\; h \in \mathbf{C}[z], \|h\|_1 \leq 1 \right\} < \infty$$

With a bit more work:

$$\mathsf{Der}(\mathrm{A}(\mathbf{D}), \mathrm{C}(\mathbf{T})/\mathrm{A}(\mathbf{D})) \cong \mathsf{P}_- \mathrm{C}(\mathbf{T}) \text{ via } D_k \leftrightarrow \partial\, \mathsf{P}_- k$$

MORE REMARKS ON THIS RESULT

- Can present the proof without using real-variable techniques.
- Instead, use an isomorphism between $\mathsf{P}_- L^\infty(\mathbf{T})$ and a space of multipliers on some Hilbert space of analytic functions.
- Most proofs of the theorem use, implicitly or explicitly, **factorization theorems** for derivatives of functions in H^p.

DAVIDSON–PAULSEN, JRAM 1997 had an operator-theoretic viewpoint, and gave a proof that gives more.

MY ORIGINAL MOTIVATION

Interested in the second Hochschild cohomology group

$$\mathsf{H}^2(\mathrm{A}(\mathbf{D}), \mathrm{A}(\mathbf{D})) = \frac{\mathsf{Z}^2(\mathrm{A}(\mathbf{D}), \mathrm{A}(\mathbf{D}))}{\mathsf{B}^2(\mathrm{A}(\mathbf{D}), \mathrm{A}(\mathbf{D}))}$$

Long known (e.g. JOHNSON, MAMS 1972) that $\mathsf{H}^2(\mathrm{A}(\mathbf{D}), \mathrm{A}(\mathbf{D})) \neq 0$, **in contrast** with what one gets for cohomology of $\mathbf{C}[z]$.

Seems that there is nothing more in the literature on $\mathsf{H}^2(\mathrm{A}(\mathbf{D}), \mathrm{A}(\mathbf{D}))\ldots$

Theorems in the rear view mirror appear closer than they are

Theorem (C., unpublished, circa 2002)

$\mathsf{H}^2(\mathrm{A}(\mathbf{D}), \mathrm{A}(\mathbf{D}))$ *is a Banach space, and there is a bounded linear map* $\psi : \mathsf{H}^2(\mathrm{A}(\mathbf{D}), \mathrm{A}(\mathbf{D})) \to \mathsf{Der}(\mathrm{A}(\mathbf{D}), \mathrm{C}(\mathbf{T})/\mathrm{A}(\mathbf{D}))$ *which is injective with dense range.*

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Question

Is the map ψ surjective?

Answers will be gratefully received!

OUTLINE OF THE PROOF

- By general theory (WHITE, PLMS 1996) or direct averaging arguments, $\mathsf{H}^n(\mathrm{A}(\mathbf{D}), L^\infty(\mathbf{T})) = 0$ for all $n \geq 1$.
- In the case $n = 2$ we can actually show $\mathsf{H}^2(\mathrm{A}(\mathbf{D}), \mathrm{C}(\mathbf{T})) = 0$ (borrowing ideas from JOHNSON, MAMS 1972) and prove that $\mathsf{B}^2(\mathrm{A}(\mathbf{D}), \mathrm{A}(\mathbf{D}))$ is **closed** in $\mathsf{Z}^2(\mathrm{A}(\mathbf{D}), \mathrm{A}(\mathbf{D}))$.
- Now use

$$\mathsf{Z}^2(\mathrm{A}(\mathbf{D}), \mathrm{A}(\mathbf{D})) \hookrightarrow \mathsf{Z}^2(\mathrm{A}(\mathbf{D}), \mathrm{C}(\mathbf{T})) = \mathsf{B}^2(\mathrm{A}(\mathbf{D}), \mathrm{C}(\mathbf{T}))$$

  and $\mathsf{Der}(\mathrm{A}(\mathbf{D}), \mathrm{C}(\mathbf{T})) = 0$ to define a continuous linear map $\mathsf{H}^2(\mathrm{A}(\mathbf{D}), \mathrm{A}(\mathbf{D})) \to \mathsf{Der}(\mathrm{A}(\mathbf{D}), \mathrm{C}(\mathbf{T})/\mathrm{A}(\mathbf{D}))$. ["Excision argument"]
- Finally, check this map is injective with dense range.

$\square$

AN $A(\mathbf{D})$-MODULE OF MULTIPLIERS

Let Λ be the measure on $\mathbf{D}$ given by $d\Lambda(z) := 4\log\frac{1}{|z|}dx\,dy$; and let

$$\mathcal{C} = \{h \in \mathcal{O}(\mathbf{D}) : \quad \|hk\|_{L^2(\Lambda)} \lesssim \|k\|_{L^2(\mathbf{T})} \quad \text{for all } k \in H^2\}.$$

Note that if $f \in H^\infty$ and $h \in \mathcal{C}$ then $fh \in \mathcal{C}$.

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Equip $\mathcal{C}$ with the natural norm, and let Ω be the closure of $\mathbf{C}[z]$ inside $\mathcal{C}$. This is a Banach A($\mathbf{D}$)-module.

Theorem (Paraphrase of known result)

Equip $\mathsf{P}_+ L^\infty(\mathbf{T})$ with the quotient norm induced from $L^\infty(\mathbf{T})$. Then $f \mapsto f'$ is a continuous bijection from $\mathsf{P}_+ L^\infty(\mathbf{T})$ onto $\mathcal{C}$.

Remark

This result can be proved by concatenating two hard results: $\mathsf{P}_+ L^\infty(\mathbf{T}) \cong \mathrm{BMOA} \cong \mathcal{C}$ where the second isomorphism is given by differentiation of functions. But there are proofs which just use complex analysis and Green's function identities.

Corollary

$\|f'\|_{\mathcal{C}} \lesssim \|f\|_\infty$ for all $f \in \mathbf{C}[z]$.

Thus $f \mapsto f'$ gives a non-zero, bounded derivation $d_\Omega : \mathrm{A}(\mathbf{D}) \to \Omega$.

NON-COMPACT DERIVATIONS?

HEATH, PhD thesis 2008 studied when various derivations on commutative Banach algebras are compact.

To my knowledge, the only recorded example of a non-compact derivation from a **uniform algebra** to a **symmetric module** is a road-runner-ish construction of J. FEINSTEIN.

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Observation (C.)

The derivation $d_\Omega : \mathrm{A}(\mathbf{D}) \to \Omega$ is not even weakly compact.

Idea of the proof. Fix $f \in H^\infty(\mathbf{D})$, then look at $(f_n) \subset \mathrm{A}(\mathbf{D})$ given by $f_n(z) = f(r_n z)$, where $r_n \nearrow 1$. If d_Ω were weakly compact we'd end up with $f' \in \Omega$. Now choose f with hindsight so this can't happen.

A universal property

Proposition (Morris, PhD thesis 1993)

If $D : \mathrm{A}(\mathbf{D}) \to M$ is a derivation and $f \in \mathrm{A}(\mathbf{D})$ then

$$\|D(f)\| \leq 2e \, \|f\|_{\mathsf{P}_+ L^\infty}$$

Corollary (Universal derivation)

Given $D : \mathrm{A}(\mathbf{D}) \to M$, there is a bounded linear map $\Omega \to M$ sending h to $h \cdot D(Z)$, and a factorization of D through $d_\Omega : \mathrm{A}(\mathbf{D}) \to \Omega$.

Remark

Any unital CBA will have a "universal symmetric module for derivations": see Runde, GMJ 1992. The point is that for $\mathrm{A}(\mathbf{D})$ we have an explicit description of a universal module.

EXPLICIT DESCRIPTION OF $\mathsf{Der}(\mathrm{A}(\mathbf{D}), \mathrm{A}(\mathbf{D})^*)$

Theorem (C.–HEATH, PAMS 2011)

Let $h \in \mathcal{O}(\mathbf{D})$, $h(0) = 0$, and suppose $h' \in H^1$. Then

$$D_h(f)(g) = \int_{\mathbf{T}} f'(z)g(z)\overline{h(z)}|dz|$$

defines a bounded derivation $D_h : \mathrm{A}(\mathbf{D}) \to \mathrm{A}(\mathbf{D})^*$, with $\|D_h\| \lesssim \|h'\|_1$. Moreover, every $D \in \mathsf{Der}(\mathrm{A}(\mathbf{D}), \mathrm{A}(\mathbf{D})^*)$ has the form D_h for a unique h as above.

Consequences

Every D_h is compact. Every D_h is 2-summing, and we can write down an **explicit** Pietsch control measure in terms of h.

Remaining problems

Determining $\mathrm{H}^2(\mathrm{A}(\mathbf{D}), \mathrm{A}(\mathbf{D}))$ explicitly

Does every derivation $D : \mathrm{A}(\mathbf{D}) \to \mathrm{C}(\mathbf{T})/\mathrm{A}(\mathbf{D})$ admit a **bounded linear lifting** $\mathrm{A}(\mathbf{D}) \to \mathrm{C}(\mathbf{T})$? Can we use dilation techniques applied to (Foguel–)Hankel operators?

Finding the universal derivation for other $R(X)$

Some partial results for X a circular domain (C. + Heath, unpublished) but nothing yet for infinitely connected domains.

Function algebras on polydiscs or Euclidean balls

Which algebras do we choose as "suitable" generalizations of $\mathrm{A}(\mathbf{D})$?

Noncommutative versions

What are the **right questions** for the NC disc algebra?