# Amenability and injectivity of locally compact quantum groups

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# Outline

Amenability & Injectivity

$\mathcal{T}(L^2(G))$-module structure of $\mathcal{B}(L^2(G))$

Quantum Group Amenability & Homology of $\mathcal{T}_\triangleright$-Modules

# Introduction

## Definition

*A locally compact group $G$ is amenable if $\exists$ a left invariant mean (LIM) on $L^\infty(G)$, i.e., a state $m \in L^\infty(G)^*$ s.t. $\langle m, l_s f\rangle = \langle m, f\rangle$.*

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**Regular Represenations**: $\lambda, \rho : G \to \mathcal{B}(L^2(G))$, defined by

$$\lambda(s)\xi(t) = \xi(s^{-1}t) \quad \text{and} \quad \rho(s)\xi(t) = \Delta(s)^{1/2}\xi(ts)$$

$$\mathcal{L}(G) = \{\lambda(s) : s \in G\}'' \quad \text{and} \quad \mathcal{R}(G) = \{\rho(s) : s \in G\}''$$

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*If $G$ is **any separable connected** locally compact group, then $\mathcal{L}(G)$ is injective.*

Example: $\mathcal{L}(SL(2,\mathbb{R}))$ is injective, but $SL(2,\mathbb{R})$ is NOT amenable.

Connection between amenability and injectivity for general $G$?

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# $\mathcal{T}(L^2(G)) \curvearrowright \mathcal{B}(L^2(G))$

## Definition (Neufang '00)

*For $\omega \in \mathcal{T}(L^2(G))$ and $T \in \mathcal{B}(L^2(G))$, define*

$$T \triangleright \omega = M_{\langle \omega, \rho(s) T \rho(s)^* \rangle} \in LUC(G) \subseteq \mathcal{B}(L^2(G)).$$

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*Given $\omega, \tau \in \mathcal{T}(L^2(G))$, define*

$$\langle \omega \triangleright \tau, T \rangle = \langle \tau, T \triangleright \omega \rangle, \quad T \in \mathcal{B}(L^2(G)).$$

# Banach Algebra Structure of $\mathcal{T}(L^2(G))$

Theorem (Neufang '00)

$(\mathcal{T}(L^2(G)), \triangleright)$ *is a* ***completely contractive Banach algebra*** *such that*

- $\pi : (\mathcal{T}(L^2(G)), \triangleright) \to (L^1(G), *)$ *is a* ***homomorphism***, *where*

$$\pi(\omega) = \omega|_{L^\infty(G)} \in L^1(G).$$

- *For any* $\omega, \tau \in \mathcal{T}(L^2(G))$,

$$\omega \triangleright \tau = \int_G \rho(s)^* \omega \rho(s) \pi(\tau)(s) ds,$$

*where* $ds$ *is the left Haar measure of* $G$.

# $LUC(G)^*$

**Completely contractive Banach algebra** structure on $LUC(G)^*$:

$$\langle m\square n, h\rangle = \langle m, n\square h\rangle \quad \text{where} \quad \langle n\square h, f\rangle = \langle n, h * f\rangle,$$

for all $m, n \in LUC(G)^*$, $h \in LUC(G)$ and $f \in L^1(G)$.

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For $m \in LUC(G)^*$, define a map $\Theta(m)$ on $\mathcal{B}(L^2(G))$ by

$$\langle \Theta(m)(T), \omega\rangle = \langle m, T \triangleright \omega\rangle, \quad T \in \mathcal{B}(L^2(G)), \omega \in \mathcal{T}(L^2(G)).$$

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Theorem (Neufang '00)

$\Theta : LUC(G)^* \to \mathcal{CB}_{\mathcal{T}_{\triangleright}}(\mathcal{B}(L^2(G)))$ *is a* ***completely isometric isomorphism of completely contractive Banach algebras***.

# Covariant Injectivity

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If $E : \mathcal{B}(L^2(G)) \to \mathcal{L}(G)$ is a covariant conditional expectation, then $E = \Theta(m)$ for some LIM $m \in LUC(G)^*$.

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### Theorem (C–Neufang '12)

*A locally compact group $G$ is amenable $\Leftrightarrow$ $\mathcal{L}(G)$ is covariantly injective.*

# Injective Modules

$\textbf{mod} - \mathcal{T}_\triangleright =$ category of right operator $(\mathcal{T}(L^2(G)), \triangleright)$-modules with completely contractive module homomorphisms.

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## Definition

*$I \in \mathbf{mod} - \mathcal{T}_\triangleright$ is injective if $\forall X, Y \in \mathbf{mod} - \mathcal{T}_\triangleright$, $\forall$ admissible injections $\Phi : X \to Y$ and $\forall$ morphisms $\Psi : X \to I$, $\exists$ morphism $\tilde{\Psi} : Y \to I$ s.t. $\tilde{\Psi} \circ \Phi = \Psi$.*

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## Theorem (C–Neufang '12)

*A locally compact group $G$ is amenable $\Leftrightarrow$ $\mathcal{L}(G)$ is injective in $\mathbf{mod} - \mathcal{T}_\triangleright$.*

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# Locally Compact Quantum Groups

## Definition (Kustermans–Vaes '00)

*A* ***LCQG*** $\mathbb{G} = (M, \Gamma, \varphi, \psi)$

- $M$ *is a* ***von Neumann algebra****;*
- $\Gamma : M \to M \bar{\otimes} M$ *is a* ***co-multiplication****: normal, unital, isometric* $*$*-homomorphism that is co-associative*

$$(\Gamma \otimes \iota) \circ \Gamma = (\iota \otimes \Gamma) \circ \Gamma;$$

- $\varphi$ *is a* ***left Haar weight*** *on* $M$*:*

$$\varphi((\omega \otimes \iota)\Gamma(x)) = \omega(1)\varphi(x), \quad x \in \mathcal{M}_{\varphi}, \ \omega \in M_*;$$

- $\psi$ *is a* ***right Haar weight*** *on* $M$*:*

$$\psi((\iota \otimes \omega)\Gamma(x)) = \omega(1)\psi(x), \quad x \in \mathcal{M}_{\psi}, \ \omega \in M_*.$$

# $L^1(\mathbb{G})$

Notation: $L^\infty(\mathbb{G}) := M$, $L^1(\mathbb{G}) := M_*$, $L^2(\mathbb{G}) := L^2(M, \varphi)$.

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**Bimodule structure** on $L^\infty(\mathbb{G})$: for $x \in L^\infty(\mathbb{G})$ and $f, g \in L^1(\mathbb{G})$,

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Definition

$LUC(\mathbb{G}) := \langle L^\infty(\mathbb{G}) \star L^1(\mathbb{G})\rangle$

# Examples

Commutative: $\mathbb{G}_a = (L^\infty(G), \Gamma_a, \varphi, \psi)$.

- $\Gamma_a(f)(s,t) = f(st)$, $\varphi$ and $\psi$ are Haar integrals.
- $(L^1(\mathbb{G}_a), \star_a) = (L^1(G), *)$.
- $LUC(\mathbb{G}_a) = LUC(G)$.

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- $LUC(\mathbb{G}_a) = LUC(G)$.

Co-commutative: $\mathbb{G}_s = (\mathcal{L}(G), \Gamma_s, \varphi)$.

- $\Gamma_s(\lambda(s)) = \lambda(s) \otimes \lambda(s)$, $\varphi = \psi$ is the Plancherel weight (e.g., $G$ discrete $\varphi(x) = \langle x\delta_e, \delta_e \rangle$).
- $(L^1(\mathbb{G}_s), \star_s) = (A(G), \cdot)$, where $A(G) = \{\langle \lambda(\cdot)\xi, \eta \rangle \mid \xi, \eta \in L^2(G)\} = \mathcal{L}(G)_*$.
- $LUC(\mathbb{G}_s) = UCB(\hat{G})$.

# Duality

**Fundamental Unitaries**: $\exists W, V \in \mathcal{B}(L^2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L^2(\mathbb{G}))$ s.t.

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**Regular Representations**: $\lambda, \rho : L^1(\mathbb{G}) \to \mathcal{B}(L^2(\mathbb{G}))$

$$\lambda(f) = (f \otimes \iota)(W) \text{ et } \rho(f) = (\iota \otimes f)(V), \ f \in L^1(\mathbb{G}).$$

**Dual Quantum Group**: $L^\infty(\hat{\mathbb{G}}) := \{\lambda(f) \mid f \in L^1(\mathbb{G})\}''$.

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Theorem (Kustermans–Vaes '00)

$\exists\ \hat{\Gamma}, \hat{\varphi}, \hat{\psi}$ *s.t.*, $\hat{\mathbb{G}} := (L^\infty(\hat{\mathbb{G}}), \hat{\Gamma}, \hat{\varphi}, \hat{\psi})$ *is a LCQG. Moreover,* $\hat{\hat{\mathbb{G}}} = \mathbb{G}$.

**Examples**: $\hat{\mathbb{G}}_a = \mathbb{G}_s$ (i.e., $\widehat{L^\infty(G)} = \mathcal{L}(G)$).

# $\mathcal{T}(L^2(\mathbb{G}))$

As $L^\infty(\mathbb{G}) \subseteq \mathcal{B}(L^2(\mathbb{G}))$, we obtain:

$$\Gamma^l : \mathcal{B}(L^2(\mathbb{G})) \ni x \mapsto W^*(1 \otimes x)W \in \mathcal{B}(L^2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L^2(\mathbb{G}))$$
$$\Gamma^r : \mathcal{B}(L^2(\mathbb{G})) \ni x \mapsto V(x \otimes 1)V^* \in \mathcal{B}(L^2(\mathbb{G}))\bar{\otimes}\mathcal{B}(L^2(\mathbb{G}))$$

Two **completely contractive Banach algebras**:

$$(\mathcal{T}(L^2(\mathbb{G})), \lhd := \Gamma^l_*) \text{ and } (\mathcal{T}(L^2(\mathbb{G})), \rhd := \Gamma^r_*)$$

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Two **completely contractive Banach algebras**:

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**Example**: $(\mathcal{T}(L^2(\mathbb{G}_a)), \triangleright_a) = (\mathcal{T}(L^2(G)), \triangleright)$.

# Module Structure on $\mathcal{B}(L^2(\mathbb{G}))$

Theorem (Hu–Neufang–Ruan '11)

$LUC(\mathbb{G}) = \langle L^\infty(\mathbb{G}) \star L^1(\mathbb{G}) \rangle = \langle \mathcal{B}(L^2(\mathbb{G})) \triangleright \mathcal{T}(L^2(\mathbb{G})) \rangle.$

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*The mapping* $\Theta : LUC(\mathbb{G})^* \ni m \mapsto \Theta(m) \in \mathcal{CB}_{\mathcal{T}_\triangleright}(\mathcal{B}(L^2(\mathbb{G})))$ *where*

$$\langle \Theta(m)(T), \omega\rangle = \langle m, T \triangleright \omega\rangle \quad T \in \mathcal{B}(L^2(\mathbb{G})), \omega \in \mathcal{T}(L^2(\mathbb{G})),$$

*is a **completely contractive representation** of* $LUC(\mathbb{G})^*$.

# Amenability

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Commutative: $\mathbb{G}_a$ is amenable $\Leftrightarrow$ $G$ is amenable.

Co-commutative: $\mathbb{G}_s$ is always amenable.

# Covariant Injectivity

## Theorem (C–Neufang '12)

*TFAE:*

1. $\mathbb{G}$ *is amenable;*
2. $\exists$ *a LIM on* $LUC(\mathbb{G})$*;*
3. $\exists$ *a covariant conditional expectation* $E : \mathcal{B}(L^2(\mathbb{G})) \to L^\infty(\hat{\mathbb{G}})$.

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**Remark**: $\mathbb{G}$ is amenable $\Leftrightarrow$ $\exists$ a conditional expectation

$E \in \mathcal{CB}^{L^\infty(\mathbb{G})}_{L^\infty(\hat{\mathbb{G}})}(\mathcal{B}(L^2(\mathbb{G})))$ (*Sołtan–Viselter '12*).

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We always have $\mathcal{CB}_{\mathcal{T}_{\triangleright}}(\mathcal{B}(L^2(\mathbb{G}))) \subseteq \mathcal{CB}^{L^\infty(\mathbb{G})}_{L^\infty(\hat{\mathbb{G}})}(\mathcal{B}(L^2(\mathbb{G})))$.

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Equality is **unknown** even for $\mathbb{G}_a = \ell^\infty(\mathbb{Z})$.

# Homological Properties of $\mathcal{T}_\triangleright$-modules

Theorem (C–Neufang '12)

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1. $\mathbb{G}$ *is amenable;*
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**Proposition (C–Neufang '12)**

*If* $\hat{\mathbb{G}}$ *is amenable then* $\mathcal{B}(L^2(\mathbb{G}))$ *is injective in* $\mathbf{mod} - \mathcal{T}_\triangleright$*.*

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*A morphism $\Phi : X \to Y$ is an* ***admissible surjection*** *if $\exists$ a completely contractive map $\Psi : Y \to X$ s.t. $\Phi \circ \Psi = \iota_Y$.*

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## Theorem (Pirkovskii '04; Kalantar–Neufang '10)

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Theorem (Ruan–Xu '97; Forrest–Hee Lee–Samei '11)

*If $G$ is an [IN]-group, then $\mathcal{L}(G)$ is injective in $A(G)-$* **mod**.

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*$\hat{\mathbb{G}}$ is amenable $\Leftrightarrow L^{\infty}(\mathbb{G})$ is injective in $\mathcal{T}_{\triangleright} - \mathbf{mod}$.*

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**Thank you!**