Finitely-generated maximal left ideals in Banach algebras

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Finitely-generated left ideals

Let A be a (complex, associative) algebra, always with an identity.

A left ideal I in A is **fg = finitely-generated** by $a_1, ..., a_n \in I$ if $I = Aa_1 + \cdots + Aa_n$.

So an algebra A is (left) Noetherian iff every left ideal is fg.

The radical of A is denoted by $J(A)$.

Now suppose that A is a Banach algebra. We investigate the **question**:

How many left ideals have to be fg to force A to be finite-dimensional?

Notation: CBA means 'commutative (unital) Banach algebra'.

Known results

Theorem (Grauert and Remmert, 1971) Let A be a CBA. Suppose that every closed ideal is fg. Then A is finite dimensional. $\square$

There was an advance by **Sinclair and Tullo** in 1974. First an easy consequence of the open mapping theorem:

Theorem Let I be a left ideal in a Banach algebra A such that $\overline{I}$ is fg. Then I is fg. $\square$

Then:

Theorem Let A be a Banach algebra. Suppose that every closed left ideal in A is fg. Then A is finite dimensional. $\square$

Question of Wieslaw Żelazko: What if we know only that all **maximal** left ideals are fg?

An easy start

(From a paper of HGD and WZ)

First this is a Banach algebra question, not a purely algebra question; consider large fields.

Let A be an infinite-dimensional Banach algebra, and consider the family of left ideals which are not fg. This family is not empty. An easy application of Zorn's lemma shows that it contains maximal elements - say these form the family $\mathfrak{M}_\infty$. (Each is closed.)

Assume that each member of $\mathfrak{M}_\infty$ is a maximal left ideal. Then we immediately have a positive answer to Żelazko's question.

However this assumption is not correct.

Example

We begin with the unital, three-dimensional algebra B which consists of the upper-triangular matrices in $\mathbb{M}_2$. Thus we identify

$$B = \mathbb{C}p \oplus \mathbb{C}q \oplus \mathbb{C}r \,,$$

where

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Define $M = \mathbb{C}p$, $I = \mathbb{C}p \oplus \mathbb{C}r$, $J = \mathbb{C}q \oplus \mathbb{C}r$. Then I and J are the two maximal left ideals in B, of codimension 1. Clearly, $M \subset I$, but $M \not\subset J$; further, $I \cap J = \mathbb{C}r$.

The identity of B is $e = p + q$; the radical of B is $J(B) = \mathbb{C}r$.

Example - continued

Take $(E, \|\cdot\|)$ to be an infinite-dimensional Banach space, and set $K = \mathbb{M}_2(E)$, so K is a unital Banach B-bimodule for 'matrix-multiplication'.

The space K satisfies $K^2 = \{0\}$, and so $K \subset J(A) = \mathbb{C}r \oplus K$.

Now $I + K$ and $J + K$ are the two maximal left ideals in A, and $(I+K) \cap (J+K) = J(A)$. The closed left ideal $M + K$ has codimension 2 in A; the only maximal left ideal that contains M is $I + K$.

We can see that $I + K = Ap + Ar$, and so it is fg. However $M + K$ is not fg: it is maximal in the above ordering, but not a maximal left ideal.

The left ideal $J + K$ is not fg, and so this example is not a counter to the conjecture. $\square$

C^*-algebras

Theorem (M. Rordam, D. Blecher - maybe well-known) Let A be a unital C^*-algebra.

(i) Suppose that I is a closed left ideal that is finitely-generated. Then $I = Ap$ for some self-adjoint projection $p \in I$.

(ii) Suppose that each maximal left ideal of A is fg. Then A is finite dimensional. $\square$

The case where $A = \mathcal{B}(E)$

Let E be a Banach space, and consider the Banach algebra $\mathcal{B}(E)$ of all bounded linear operators on E. For 'most', maybe all, Banach spaces E, the conjecture holds.

See the next talk of Tomek Kania.

The case where $A = L^1(G)$

For 'most', maybe all, locally compact groups G, the conjecture holds.

Commutative Banach algebras

Fact Assume that $M \in \mathfrak{M}_\infty$ is actually an ideal. Then it is a maximal left ideal, and so the conjecture follows. Thus it holds for CBAs.

We can do more.

Let A be a CBA. Then the Gelfand transform is a homomorphism from A onto $\widehat{A} \subset C(X)$, where X is the character space or maximal ideal space of A. Each maximal ideal is

$$M_x = \{a \in A : \widehat{a}(x) = 0\}$$

for $x \in X$.

The **Shilov boundary** of A is the minimum closed set Γ in X such that $|f|_\Gamma = |f|_X$ for all $f \in \widehat{A}$ (where $|\cdot|_X$ is the uniform norm on X).

Triviality: Suppose that x is an isolated point in X. Then there exists $\chi \in M_x$ such that $\chi(y) = 1$ $(y \neq x)$, and so M_x is singly generated by χ. The converse does not hold in general (cf. disc algebra). However:

Boundaries

Theorem Let A be a CBA, and take $x \in \Gamma(A)$. Then the following are equivalent:

(a) M_x is singly generated;

(b) M_x is fg;

(c) x is isolated.

For (b) $\Rightarrow$ (c), use Gleason's argument to put a copy of an analytic variety (in $\mathbb{C}^n$) around x and use the maximum modulus principle for holomorphic functions on varieties. $\square$

Thus:

Theorem Let A be a CBA. Suppose that M_x is fg for each $x \in \Gamma$. Then A is finite dimensional. $\square$

Peak points

There is a smaller boundary than the Shilov boundary; this is the **Choquet boundary**, Γ_0. For X metrizable, Γ_0 consists of the **peak points** :

Definition A point $x \in X$ is a **peak point** (for A) if there exists $f \in \widehat{A}$ such that $f(x) = 1$ and $|f(y)| < 1 \ (y \neq x)$.

By Shilov's idempotent theorem, each isolated point of X is a peak point.

Is it sufficient that M_x be fg for each peak point?

Uniform algebras

Theorem Let A be a uniform algebra on a (metrizable) space X, and suppose that M_x is fg for each peak point x. Then A is finite dimensional. $\square$

Proof Each peak point is isolated, and so $\Gamma_0 = \{\varphi_n : n \in \mathbb{N}\}$ is open, with compact complement, say L. Let δ_n peak at φ_n. Assume $L \neq \emptyset$. Then

$$1 - \sum_{n=1}^{\infty} \frac{1}{n} \delta_n$$

peaks on L. But, for uniform algebras, every peak set contains a peak point, contradiction. $\square$

But the answer is 'no' for more general CBA:

An example - quasi-analytic algebras

Start with the closed unit disc $\overline{\mathbb{D}}$.

Put a **quasi-analytic Banach function algebra** on $\overline{\mathbb{D}}$. Thus A is a Banach algebra of infinitely-differentiable functions with the properties that the character space of A is $\overline{\mathbb{D}}$ and, for each $z \in \overline{\mathbb{D}}$ and each $f \in A$ with $f \neq 0$, there exists $n \in \mathbb{N}$ with $f^{(n)}(z) \neq 0$.

For example, take the norm to be

$$\|f\| = \sum_{k=0}^{\infty} \frac{\left|f^{(k)}\right|_{\overline{\mathbb{D}}}}{M_k},$$

where $M_k = k! \log 3 \cdots \log(k+3) \ (k \in \mathbb{N})$.

The uniform closure of B is the disc algebra.

An example - Lipschitz algebras

Add n points equally spaced on the disc of radius $1+1/n$ for each $n \in \mathbb{N}$, to form the set U. Take $L = \mathbb{T} \cup U$, and put an algebra C of Lipschitz functions on L. Thus C consists of the continuous functions on L such that $\|\cdot\|_C < \infty$, where $\|\cdot\|_C$ is specified by the formula

$$\|h\|_C = |h|_L + \sup\left\{\frac{|h(z)-h(w)|}{|z-w|} : z, w \in L,\, z \neq w\right\}.$$

It is easy that C is a natural Banach function algebra on L.

The example

Combine these Banach function algebras to form a Banach function algebra, called A, on $K = \overline{\mathbb{D}} \cup U$.

Let $\mathfrak{A}$ be the subalgebra of A consisting of the functions $f \in A$ on X with

$$\lim_{n\to\infty} \frac{f(x_n) - f(z_0)}{x_n - z_0} = f'(z_0)$$

whenever $z_0 \in \mathbb{T}$ and (x_n) is a sequence in U with $\lim_{n\to\infty} x_n = z_0$.

We can check that $\mathfrak{A}$ is closed in A and that it is a natural Banach function algebra on K.

The Choquet boundary

For this example, each point x of U is isolated and so a peak point (and M_x is singly generated). But we **claim** that no point of $\overline{\mathbb{D}}$ is a peak point!

Assume that $f \in \mathfrak{A}$ and that f peaks at 1, say with $f(1) = 1$. Then $f \mid \overline{\mathbb{D}} = 1 + g$ for some $g \in B$. The function g is not zero, and so, since B is a quasi-analytic algebra, there exists $k \in \mathbb{N}$ such that $g^{(k)}(1) \neq 0$; we take $k_0 \in \mathbb{N}$ to be the minimum such k, so that there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that

$$f(z) = 1 + \alpha(z-1)^{k_0} + o(|z-1|^{k_0})$$

as $z \to 1$ in K.

The Choquet boundary - continued

First, suppose that $k_0 \geq 2$. Then it is easy to find a sequence (z_n) in $\overline{\mathbb{D}}$ with $|f(z_n)| > 1$ for all sufficiently large $n \in \mathbb{N}$, a contradiction.

Second, suppose that $k_0 = 1$. We may suppose that $\alpha > 0$. Now we can find a sequence (x_n) in U with $\Re(\alpha(x_n - 1)) > 0$ for each n. But

$$\lim_{n\to\infty} \Re\left(\frac{f(x_n) - f(1)}{x_n - 1}\right) = \alpha > 0\,.$$

Thus $\Re f(x_n) > 1$ for all sufficiently large n, again a contradiction.

Thus $\Gamma_0 = U$, but $\Gamma = L$.

The algebra $\mathfrak{A}$ is not finite dimensional. $\square$

References

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