THE CHOQUET BOUNDARY OF AN OPERATOR SYSTEM

Kenneth R. Davidson

University of Waterloo

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joint work with Matthew Kennedy
I would like to dedicate this talk to

Bill Bade (1924–2012)

and

Bill Arveson (1934–2011).
B. Sz.Nagy began an extensive development of dilation theory. With Foiaş it became a key tool for studying a single operator.
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**Theorem (Sz. Nagy (1953))**

If $T \in B(\mathcal{H})$ and $\|T\| \leq 1$, there is a unitary operator of form

$$ U = \begin{bmatrix} * & 0 & 0 \\ * & T & 0 \\ * & * & * \end{bmatrix} $$
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**Corollary (Generalized von Neumann inequality)**

If $[p_{ij}]$ is a matrix of polynomials, and $\|T\| \leq 1$, then

$$\| [p_{ij}(T)] \| \leq \sup_{|z| \leq 1} \| [p_{ij}(z)] \|.$$
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**Theorem (Sz. Nagy (1953))**

If $T \in \mathcal{B}(\mathcal{H})$ and $\|T\| \leq 1$, there is a unitary operator of form

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Hence this can be considered as a study of representations of the disk algebra $A(\mathbb{D})$. 
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The main themes of his approach were:

- **Operator algebra** $\mathcal{A}$: unital subalgebra of a C*-algebra $C^*(\mathcal{A})$.
  Hence: a norm structure on matrices $\mathcal{M}_n(\mathcal{A}) \subset \mathcal{M}_n(C^*(\mathcal{A}))$. 

The role of completely positive and completely bounded maps.

$\phi: \mathcal{A} \to \mathcal{B}(\mathcal{H})$ induces

$\phi_n : \mathcal{M}_n(\mathcal{A}) \to \mathcal{M}_n(\mathcal{B}(\mathcal{H})) \cong \mathcal{B}(\mathcal{H}^n)$

by $\phi_n ([a_{ij}]) = [\phi(a_{ij})]$.

Say $\phi$ is completely bounded (c.b.) if

$\|\phi\|_{cb} = \sup_{n \geq 1} \|\phi_n\| < \infty$.

Say $\phi$ is completely contractive (c.c.) if

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Operator system $S$: unital s.a. subspace $1 \in S = S^* \subset C^*(S)$. 

If $\phi: S \to B(H)$, then $\phi$ is completely positive (c.p.) if $\phi^n$ is positive for all $n \geq 1$. Say $\phi$ is u.c.p. if also $\phi(1) = I$. If $\rho: A \to B(H)$ is a c.c. unital map, then $S = A + A^*$ and $\tilde{\rho}(a + b^*) = \rho(a) + \rho(b^*)$ is a u.c.p. extension to $S$. 

Theorem (Arveson's Extension Theorem): If $\phi: S \to B(H)$ is c.p. and $S \subset T$, then there is a c.p. map $\psi: T \to B(H)$ s.t. $\psi|_S = \phi$. i.e. $B(H)$ is injective.
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• If $\rho : A \to B(H)$ is a c.c. unital map, then $S = \overline{A + A^*}$ and

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Ken Davidson and Matt Kennedy

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**Theorem (Arveson’s Extension Theorem)**

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A dilation of a u.c.c. representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a u.c.c. representation $\sigma : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ where $\mathcal{K} = \mathcal{K}_- \oplus \mathcal{H} \oplus \mathcal{K}_+$, and

$$\sigma(a) = \begin{bmatrix} * & 0 & 0 \\ * & \rho(a) & 0 \\ * & * & * \end{bmatrix}.$$
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$$\psi(a) = \begin{bmatrix} \varphi(a) & * \\ * & * \end{bmatrix}.$$

Note that if $\sigma \succ \rho$, then $\tilde{\sigma} \succ \tilde{\rho}$.
But $\psi \succ \tilde{\rho}$ may not be multiplicative on $\mathcal{A}$.
**Theorem (Arveson’s Dilation Theorem)**

Let \( \rho : A \to \mathcal{B}(\mathcal{H}) \) be a representation. TFAE

1. \( \rho \) is u.c.c.
2. \( \bar{\rho} \) is u.c.p.
3. \( \rho \) dilates to a unital \(*\)-representation of \( C^*(A) \).
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Now we turn to two central ideas in Arveson’s paper which he was not able to verify in general:

- boundary representations
- the $\mathbb{C}^*$-envelope
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Now we turn to two central ideas in Arveson’s paper which he was not able to verify in general:

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Bill was able to verify this in many concrete examples. See also Subalgebras of $C^*$-algebras II, Acta Math. 128 (1972), 271–308.
A u.c.p. map \( \varphi : S \to B(\mathcal{H}) \) or a u.c.c. repn. \( \varphi : A \to B(\mathcal{H}) \) has the unique extension property (u.e.p) if

1. \( \varphi \) has a unique u.c.p. extension to \( C^*(S) \) (or \( C^*(A) \)), and
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If $1 \in A \subset C(X)$, then irreducible repns. are point evaluations $\delta_x$. A u.c.p. extension is given by a **representing measure** $\mu$ on $X$:

$$f(x) = \int_X f \, d\mu \quad \text{for all} \quad f \in A.$$
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$\iff$ $x$ has a unique representing measure

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The boundary representations form the Choquet boundary of $S$. 
The C*-envelope of $\mathcal{A}$ (or $\mathcal{S}$) is a pair $(C^*_{\text{env}}(\mathcal{A}), \iota)$ where
\[ \iota : \mathcal{A} \to C^*_{\text{env}}(\mathcal{A}) \text{ is comp. isom. iso., } C^*_{\text{env}}(\mathcal{A}) = C^*(\iota(\mathcal{A})), \]
with universal property: if $j : \mathcal{A} \to \mathcal{B} = C^*(j(\mathcal{A}))$ comp. isom. iso.
then $\exists q : \mathcal{B} \to C^*_{\text{env}}(\mathcal{A})$ *-homomorphism s.t. $q j = \iota$.

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\iota} & C^*(\iota(\mathcal{A})) \\
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with universal property: if $j : \mathcal{A} \rightarrow \mathcal{B} = C^*(j(\mathcal{A}))$ comp. isom. iso. 
then $\exists q : \mathcal{B} \rightarrow C^*_\text{env}(\mathcal{A})$ $\ast$-homomorphism s.t. $qj = \iota$.

If there are sufficiently many boundary representations $\{\pi_\lambda\}$ 
to completely norm $\mathcal{A}$ or $\mathcal{S}$, let $\pi = \bigoplus \pi_\lambda$. Then 
$C^*_\text{env}(\mathcal{S}) = C^*(\pi(\mathcal{S}))$. 
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**Theorem (Hamana (1979))**

Every operator system is contained in a unique minimal injective operator system.

**Corollary (Hamana)**

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Muhly-Solel (1998) gave a homological characterization of boundary representations.
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- if \( \rho \) is a c.i.i., and \( \sigma \succ \rho \) is maximal, then
  \[
  C^*_{\text{env}}(\mathcal{A}) = C^*(\sigma(\mathcal{A})).
  \]
The next four decades

Our approach

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This dilation proof yields important information about $C^*_{env}(A)$. It does not yield boundary representations.
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Muhly-Solel result says: a repn. has u.e.p. $\iff$ it is an extremal extension and an extremal coextension.
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- reworks Dritschel-McCullough for operator systems
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**Theorem (Arveson (JAMS 2008))**

If $S$ is separable, then there are sufficiently many boundary representations.
Our approach

- We give a dilation theory proof of the existence of boundary representations.
- It works in complete generality.
- The argument is conceptual and natural.
A c.p. map $\varphi$ is pure if $0 \leq \psi \leq \varphi$ implies $\psi = t\varphi$. 

Arveson (1969) If $\pi$ is reducible, then there exists $P = P^2 \in \pi(S)'$. Then $\psi(a) = P\varphi(a)$ satisfies $0 \leq \psi \leq \varphi$ but $\psi(1) = P \neq tI = \varphi(1)$. So $\pi$ is a boundary repn.
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Arveson (2008) Say $\varphi$ is maximal at $(s, x)$ if

$$\psi \succ \varphi \implies \|\psi(s)x\| = \|\varphi(s)x\|.$$

If $\varphi$ is maximal at every $(s, x)$, then $\varphi$ is maximal.
**Key Lemma**

If \( \varphi \) is pure, and \((s_0, x_0) \in S \times \mathcal{H} \), then there is a pure dilation \( \psi : S \to \mathcal{B}(\mathcal{H} \oplus \mathbb{C}) \) s.t. \( \psi \succ \varphi \) and \( \psi \) is maximal at \((s_0, x_0)\).
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- If \( \psi : S \to \mathcal{B}(\mathcal{H} \oplus K) \), then compression to \( \text{span}\{\mathcal{H}, \psi(s_0)x_0\} \) has same norm at \((s_0, x_0)\).
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- \( \{\psi : S \to B(\mathcal{H} \oplus \mathbb{C}) : \psi \succ \varphi\} \) is BW-compact.
  Hence \( \exists \psi \) s.t. \( \psi(s_0)x_0 = \varphi(s_0)x_0 \oplus \eta \) with \( \eta \) maximal.
Key Lemma

If \( \varphi \) is pure, and \((s_0, x_0) \in S \times \mathcal{H}\), then there is a pure dilation \( \psi : S \to B(\mathcal{H} \oplus \mathbb{C}) \) s.t. \( \psi \succ \varphi \) and \( \psi \) is maximal at \((s_0, x_0)\).

- If \( \psi : S \to B(\mathcal{H} \oplus \mathcal{K}) \), then compression to \( \text{span}\{\mathcal{H}, \psi(s_0)x_0\} \) has same norm at \((s_0, x_0)\).
- \( \{\psi : S \to B(\mathcal{H} \oplus \mathbb{C}) : \psi \succ \varphi\} \) is BW-compact.
  Hence \( \exists \psi \) s.t. \( \psi(s_0)x_0 = \varphi(s_0)x_0 \oplus \eta \) with \( \eta \) maximal.
- Take extreme point \( \psi_0 \) of
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- Delicate argument to show that \( \psi_0 \) is pure.
Theorem 1

Every pure u.c.p. map $\varphi : S \to \mathcal{B}(\mathcal{H})$ dilates to a maximal pure u.c.p. map, and hence extends to a boundary representation.
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Every pure u.c.p. map \( \varphi : S \to B(\mathcal{H}) \) dilates to a maximal pure u.c.p. map, and hence extends to a boundary representation.

- routine transfinite induction to obtain dilation maximal at every pair \((s, x)\)
- if \( S \) is separable and \( \dim \mathcal{H} < \infty \), then can produce the maximal dilation as limit of sequence of finite dim. maps.
**Theorem 2**

*There are sufficiently many boundary representations to completely norm $S$.***
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- Dilate it to a boundary repn. $\sigma$ of $\mathcal{M}_n(S)$ by Theorem 1. Then $\sigma \simeq \pi^{(n)}$, where $\pi$ is irreducible repn. of $\mathcal{C}^*(S)$.
- If $\varphi$ is u.c.p. dilation of $\pi|_S$, then $\varphi^{(n)}$ dilates $\sigma|_{\mathcal{M}_n(S)}$. Hence $\varphi = \pi$. So $\pi$ is the desired boundary repn. (This is easy direction of a result of Hopenwasser.)
Second method to get sufficiently many boundary repns. A **matrix state** is a u.c.p. map of $S$ into $\mathcal{M}_n$.

**Theorem**

The pure matrix states completely norm $S$. 

**THEOREM**

Finite dimensional compressions of a faithful repn. of $C^* (S)$ completely norm $S$. So matrix states completely norm $S$.

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**Theorem**

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- Finite dimensional compressions of a faithful repn. of $\mathcal{C}^*(S)$ completely norm $S$. So matrix states completely norm $S$.
- The collection of all matrix states $(S_n(S))_{n \geq 1}$ is $\mathcal{C}^*$-convex: If $\gamma_j \in \mathcal{M}_{n_j,n}$, $\sum_{j=1}^{k} \gamma_j^* \gamma_j = I_n$ and $\psi_j \in S_{n_j}(S)$, then
  \[
  \psi = \sum_{j=1}^{k} \gamma_j^* \psi_j \gamma_j \in S_n(S).
  \]
  Can define $\mathcal{C}^*$-convex hull.
There is a notion of \textit{C*-extreme point} of a C*-convex set.

\textbf{Farenick (2000)} shows that the C*-extreme points of \((S_n(S))_{n \geq 1}\) coincide with the pure matrix states.
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**Theorem (Farenick 2004)**

The $C^*$-convex hull of the pure matrix states is BW-dense in the set of all matrix states.
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**Theorem (Farenick 2004)**

The C*-convex hull of the pure matrix states is BW-dense in the set of all matrix states.

Hence the pure matrix states completely norm S.
Putting it all together, we obtain:

**Theorem 3**

Every operator system and every unital operator algebra has sufficiently many boundary representations.
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**Theorem 3**

*Every operator system and every unital operator algebra has sufficiently many boundary representations.*

**Corollary**

The C*-envelope of every operator system and every unital operator algebra is obtained from a direct sum of boundary representations.
Where does this get us?

- Over four decades, we developed many techniques to get our hands on the C*-envelope of an operator algebra without using boundary representations.
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- The Choquet boundary, peak points and representing measures play a central role in the study of function algebras.
- Perhaps now, we can more diligently pursue the use of boundary representations in non-commutative dilation theory. This was central to Arveson’s vision of the subject.
Our paper is available on the arXiv:1303.3252

- K.R. Davidson and M. Kennedy,
  
  *The Choquet boundary of an operator system.*
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I wish to draw your attention to two recent surveys of Bill Arveson’s work in JOT:

The end.

Tack.