The Eberlein Compactification of the Heisenberg Type Group $\mathbb{Z} \times \mathbb{T} \times \mathbb{T}$

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Outline

- The group $H = \mathbb{Z} \times \mathbb{T} \times \mathbb{T}$
- Semigroup Compactifications
  - Definition
  - Eberlein Compactification, H^e
- Structure of H^e in terms of $(\mathbb{Z} \times \mathbb{T})^e$ and $\mathbb{T}$
- Structure of $(\mathbb{Z} \times \mathbb{T})^e$ in terms of $\mathbb{Z}^e$ and $\mathbb{T}$
- "Structure" of $\mathbb{Z}^e$
  - Weakly Almost Periodic Compactification

$\mathbb{Z} \times \mathbb{T} \times \mathbb{T}$

- We will consider $(\mathbb{Z}, +)$ and $(\mathbb{T}, .)$.
- On $H = \mathbb{Z} \times \mathbb{T} \times \mathbb{T}$, *Heisenberg type multiplication*:

$$(n, x, s)(m, y, t) = (n + m, xy, sy^n t).$$

- $e = (0, 1, 1)$ and $(n, x, y)^{-1} = (-n, x^{-1}, y^{-1}x^n)$.
- H is a locally compact group with *product topology*.
- $\mathbb{T}$ is a normal subgroup and $Z(H) = \mathbb{T}$.
- $\mathbb{Z} \times \mathbb{T}$ is not a subgroup of H, but $H/\mathbb{T} \cong \mathbb{Z} \times \mathbb{T}$.

Goal:

- Determine the structure of the Eberlein compactification of $H = \mathbb{Z} \times \mathbb{T} \times \mathbb{T}$, denoted by H^e.
- That is, we want to construct a compact semigroup, H^e,
  - H is dense in H^e,
  - Uniform limits of Fourier-Stieltjes functions of H extend continuously to H^e.

G: locally compact group.

- (ψ, X) is a *semigroup compactification* if
  - X is a compact, Hausdorff, right topological semigroup;
  - $\psi : G \to X$ is a continuous homomorphism;
  - $\psi(G)$ is dense in X;
  - $\psi(G)$ is in the *topological center*
    $\Lambda(X) = \{t \in X : \ X \to X : s \to ts \text{ is continuous}\}$.
- Identify G with $\psi(G)$.
- Classify compactifications wrt:
  - algebraic/ topological properties of X
    e.g. X: topological group, semitopological semigroup.
  - properties of $\mathcal{C}(X)|_{\psi(G)} = \{f \circ \psi : f \in \mathcal{C}(X)\} \subset \mathcal{C}_b(G)$
    e.g. $\mathcal{C}(X)|_{\psi(G)} \subset AP(G)$ or $\mathcal{C}_0(G) + \mathbb{C}$.

- **Thm.** If $\mathcal{A} \subset \mathcal{C}_b(G)$ is
  - a subalgebra, norm closed, conjugate closed
  - translation invariant, contains the constants
  - invariant under introversion operators

  $(\epsilon, \sigma(\mathcal{A}))$ gives a semigroup compactification of G with $\mathcal{C}(\sigma(\mathcal{A}))|_G = \mathcal{A}$.

More generally,

- If $\mathcal{C}(X)|_G \subset \mathcal{A}$, then X is an $\mathcal{A}$-compactification.
- If $\mathcal{C}(X)|_G = \mathcal{A}$, then X is the Universal $\mathcal{A}$-compactification, denoted by G^a;

B(G), Fourier-Stieltjes Algebra

- $\mathcal{U}(\mathcal{H})$: *unitary operators* on Hilbert sp $\mathcal{H}$
- A *unitary representation* of G

$$\pi : G \to \mathcal{U}(\mathcal{H})$$

  continuous wrt *SOT*.

- $\Sigma_G = \{$Equivalence classes of unitary repn's of $G\}$.
- $B(G) = \{g \to \langle \pi(g)\xi, \eta \rangle : \pi \in \Sigma_G, \xi, \eta \in \mathcal{H}_\pi\}$
  - $B(G)$ is a subalgebra of $\mathcal{C}_b(G)$

$\mathcal{E}(G)$, Eberlein Algebra

$B(G)$ satisfies the criteria of de *Thm*, but $B(G)$ is NOT uniformly closed.

- $\mathcal{E}(G) = \overline{B(G)}^{\|\cdot\|_\infty}$,
- $C_0(G) \subset \mathcal{E}(G) \subset WAP(G) \subset UC(G)$.

- The corresponding compactification is denoted by (ϕ, G^e).
- G^e is a semitopological semigroup.
- ϕ is a homeomorphism.
- [Megrelishvili 2008, Spronk & Stokke 2011]
  G^e is universal amongst all compactifications of G which can be embedded as contractions on a Hilbert space.

$(\mathbb{Z} \times \mathbb{T} \times \mathbb{T})^e$

As a consequence of the definition of H^e:

- Let the Eberlein compactification of $\mathbb{Z} \times \mathbb{T}$ be

$$(\mu, (\mathbb{Z} \times \mathbb{T})^e)$$

- $\mu \circ \eta : H \to H/\mathbb{T} \cong \mathbb{Z} \times \mathbb{T} \to (\mathbb{Z} \times \mathbb{T})^e$,
- $(\mathbb{Z} \times \mathbb{T})^e$ is a quotient of H^e.

- $\mathbb{T}^e = \mathbb{T}$,
- $Z(H) = \mathbb{T}$ implies that $\mathcal{E}(\mathbb{T}) = \mathcal{C}(\mathbb{T}) = \mathcal{E}(H)|_{\mathbb{T}}$.
- Hence, $\mathbb{T}$ is a quotient of H^e.

- Goal: To study H^e, understand the structure of $\mathcal{E}(H)$.
- Let $f \in \mathcal{E}(H)$, consider

$$f(n,x,y) = \underbrace{f(n,x,1)} + \underbrace{f(n,x,y) - f(n,x,1)}.$$

- The function $h(n,x) = f(n,x,1)$ is in $\mathcal{E}(\mathbb{Z} \times \mathbb{T})$.
- We claim that $g(n,x,y) = f(n,x,y) - f(n,x,1)$ is in $\mathcal{C}_0(H)$.
  - **Lemma:**

$$\lim_{n\to\infty} \sup\{|f(n,x,y) - f(n,x,y')| \mid y, y', x \in \mathbb{T}\} = 0. \qquad (1)$$

Proof of (1):

- For $\varepsilon > 0$, by uniform continuity of f, there is a compact nbhd W of e:
  - $W = V \times U \times O$,
    - V compact nbhd of 0,
    - U, O compact nbhds of 1;
  - $(n', x', y')(n, x, y)^{-1} \in W$ or $(n, x, y)^{-1}(n', x', y') \in W$ means

$$| f(n, x, y) - f(n', x', y') | < \varepsilon \qquad (2)$$

- Fix $x \in \mathbb{T}$ and $y, y' \in \mathbb{T}$.

- For $x \in \mathbb{T} \Rightarrow \exists N \in \mathbb{Z}$ s.t. $\forall m \in \mathbb{Z}$, $\mid m \mid > N$
$$\{z^m | 0 \leq arg(z) \leq arg(x)\} = \mathbb{T}.$$
- Choose $z \in U$ and m large enough such that
$$z^m = y' y^{-1}.$$
- Then
$$(m, x, y)(m, xz^{-1}, y)^{-1} = (0, z, 1) \in W$$
and
$$(m, xz^{-1}, y)^{-1}(m, x, y') = (0, z, 1) \in W$$
- Hence, by (2),
$$|f(m, x, y) - f(m, x, y')|$$
$$\leq |f(m, x, y) - f(m, xz^{-1}, y)| + |f(m, xz^{-1}, y) - f(m, x, y')|$$
$$< 2\varepsilon$$

Consequences:

- $\mathcal{E}(H) \cong \mathcal{E}(\mathbb{Z} \times \mathbb{T}) + \mathcal{C}_0(H)$.
  - It is not a direct sum, since $\mathcal{E}(\mathbb{Z} \times \mathbb{T}) \cap \mathcal{C}_0(H) \neq \{0\}$.
  - Topologically,

$$(\mathbb{Z} \times \mathbb{T} \times \mathbb{T})^e = H \sqcup ((\mathbb{Z} \times \mathbb{T})^e \setminus (\mathbb{Z} \times \mathbb{T})).$$

  - $(\mathbb{Z} \times \mathbb{T} \times \mathbb{T})^e$ is obtained from $(\mathbb{Z} \times \mathbb{T})^e$ by adjoining $\mathbb{T}$, with a different topology.

More generally:

- Consider $G = H_1 \times H_2 \times N$ equipped with Heisenberg type structure.
- We say G satisfies the *small transitivity condition on* H_i if: for any $n \in N$, any compact neighborhood V of 0_2 in H_j, $i \neq j$, there exists a compact subset C of H_i such that for any $x \in H_i \setminus C$, we have $\varphi(x, V) \cap nO \neq \emptyset$ for any neighborhood O of 1 in N.
- In this case, $\mathcal{E}(G) = \mathcal{E}(H_1 \times H_2) + \mathcal{C}_0(G)$
- **Examples:**
  $G = H_1 \times H_2 \times N = \mathbb{Z} \times H \times H$, where H is a connected compact Abelian group.
  $G = \mathbb{R} \times \mathbb{R} \times \mathbb{T}$ [Milnes 1981]

$(\mathbb{Z} \times \mathbb{T})^e$

Goal. Construct $(\mathbb{Z} \times \mathbb{T})^e$ from $\mathbb{Z}^e$ by an idea of Hahn [1960].

- Let $Q = \mathbb{Z} \times \mathbb{T}$. On the direct product $Q \times \mathbb{Z}^e$,

$$(x, s)\rho(y, t) \text{ iff } y^{-1}x \in \mathbb{Z} \text{ and } (y^{-1}x)s = t$$

- $(Q \times \mathbb{Z}^e)/\rho$ is a compact stpl sgr with quotient mapping of

$$(x, s)(y, t) = (xy, st)$$

- The mapping $\mu : Q \to (Q \times \mathbb{Z}^e)/\rho$, where

$$\mu(x) = [(x, 1)]$$

is a homeomorphism onto a dense subset.

- $(\mu, (Q \times \mathbb{Z}^e)/\rho)$ satisfies the following universal property:
  - Let (φ, X) be a semigroup compactification of Q such that
    - $\varphi|_{\mathbb{Z}}$ extends to a continuous homomorphism $\tilde{\varphi} : \mathbb{Z}^e \to X$.
    - For each $x \in Q$ and $s \in \mathbb{Z}^e$

$$\tilde{\varphi}(x^{-1}sx) = \varphi(x^{-1})\tilde{\varphi}(s)\varphi(x).$$

  Then there is a unique homomorphism $\vartheta : (Q \times \mathbb{Z}^e)/\rho \to X$ such that

$$\vartheta \circ \mu = \varphi$$

- $(\mu, (Q \times \mathbb{Z}^e)/\rho)$ is the Eberlein compactification of $Q = \mathbb{Z} \times \mathbb{T}$.

Consequences on $(\mathbb{Z} \times \mathbb{T})^e$

$$(\mathbb{Z} \times \mathbb{T})^e \cong (\mathbb{Z} \times \mathbb{T} \times \mathbb{Z}^e)/\rho$$

- Consider the restriction of ρ to $\mathbb{T} \times \mathbb{Z}^e$:

$$(x, s)\rho(y, t) \Leftrightarrow \quad y = x \text{ and } s = t.$$

- $(\mathbb{Z} \times \mathbb{T})^e$ is obtained from $\mathbb{Z}^e$ by adjoining a copy of $\mathbb{T}$ for each element of $\mathbb{Z}^e$.
- $(\mathbb{Z} \times \mathbb{T})^e$ contains uncountably many copies of $\mathbb{T}$.

To determine the structure of $(\mathbb{Z} \times \mathbb{T} \times \mathbb{T})^e$ we need to understand $\mathbb{Z}^e$.

Structure of $\mathbb{Z}^e$

- $\mathbb{Z}^e$ is a compact semitopological semigroup.
- $\mathbb{Z}^e$ has been studied as a quotient of the largest semitopological semigroup compactification, $\mathbb{Z}^w$ of $\mathbb{Z}$, by [Brown, Moran, Bouziad, Lemanczyk, Mentzen, et al.]
- $\mathbb{Z}^e$ is a proper quotient of $\mathbb{Z}^w$.
- $\mathbb{Z}^e$ mimic the structure of $\mathbb{Z}^w$ in many ways.
- Questions about sizes of the differences between $\mathbb{Z}^e$ and $\mathbb{Z}^w$ remain unknown.

$WAP(G)$

- Let $f \in \mathcal{C}_b(G)$. The *orbit* of f is defined by

$$O(f) = \{f_g(\cdot) = f(g\cdot) : g \in G\},$$

- We call f a **weakly almost periodic function** (**w.a.p**) if the $O(f)$ is relatively weakly compact.

- $WAP(G)$ denotes the set of w.a.p functions on G.

Properties of G^w

- w.a.p compactification G^w exists.
- G^w is universal among (ψ, X) s.t. X is semitopological.
- $\cdot^{-1} : G \to G : g \mapsto g^{-1}$ extends to a cts involution on G^w.
- G^w is the universal involutive compactification.
- [Shtern 1993, Megrelishvili 2007]
  G^w is universal amongst all compactifications of G which representable as unif. bdd. lin. transf. on ref. Banach Sps.
- $\mathbb{C} + \mathcal{C}_0(G) \subset WAP(G)$
- $\psi : G \to G^w$ is a homeomorphism.

$\mathbb{Z}^e$

- $\mathbb{Z}^e$ (and hence $\mathbb{Z}^w$) contains countably many copies of $B_1(L^\infty([0,1]))$.
- The set of idempotents of both $\mathbb{Z}^e$ and $\mathbb{Z}^w$ are not closed.
- $\mathbb{Z}^e$ has uncountably many idempotents.
- *Question:* Cardinality of the set of idempotents of $\mathbb{Z}^e$?
- [Ruppert 1991] Cardinality of idempotents of $\mathbb{Z}^w$ is 2^c.

Thank You!