

Nonself-adjoint 2-graph Algebras

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Row-isometries

Let S_1, \dots, S_n be isometries on \mathcal{H} with pairwise orthogonal ranges, i.e.

$$S_i^* S_j = \delta_{i,j} I.$$

Then $S = [S_1, \dots, S_n]$ is a row-isometry, i.e. is an isometric map from $\mathcal{H}^{(n)}$ to \mathcal{H} .

Conversely an isometric map from $\mathcal{H}^{(n)}$ is determined by n isometries on \mathcal{H} with pairwise orthogonal ranges.

We say a row-isometry is of *Cuntz-type* if

$$\sum_{i=1}^n S_i S_i^* = I.$$

We will be interested in “commuting” row-isometries and the algebras they generate.

Let $S = [S_1, \dots, S_n]$ be a Cuntz-type row-isometry. Then

- 1 there is only one possible C^* -algebra (Cuntz),
- 2 there is only one possible unital norm-closed algebra (Popescu),
- 3 the weak operator closed unital nonself-adjoint algebras are determined by the structure of the row-isometry (Davidson-Katsoulis-Pitts; Kennedy).

Representations of single vertex 2-graphs

Let $S = [S_1, \dots, S_m]$ and $T = [T_1, \dots, T_n]$ be row-isometries on \mathcal{H} and let θ be a permutation on $m \times n$ elements. Then S and T are θ -commuting row-isometries if

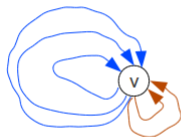
$$S_i T_j = T_{j'} S_{i'} \text{ when } \theta(i, j) = (i', j').$$

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This is precisely saying that (S, T) is an isometric representation of the 2-graph



An important example: the left-regular representation

Let $\mathcal{H}_n = \ell^2(\mathbb{F}_n^+)$ with orthonormal basis $\{\xi_w : w \in \mathbb{F}_n^+\}$. Define the row-isometry $L = [L_1, \dots, L_n]$ by

$$L_i \xi_w = \xi_{iw}.$$

Let $\mathcal{A}_n = \overline{\text{alg}}^{\|\cdot\|} \{I, L_1, \dots, L_n\}$. We call this the *noncommutative disc algebra*. (Note when $n = 1$, $\mathcal{A}_1 = A(\mathbb{D})$).

Let $\mathfrak{L}_n = \overline{\text{alg}}^{\text{WOT}} \{I, L_1, \dots, L_n\}$. We call this the *noncommutative analytic Toeplitz algebra*. (Note when $n = 1$, $\mathfrak{L}_1 = H^\infty$).

An important example: the left-regular representation

Let θ be a permutation on $m \times n$ and let \mathbb{F}_θ^+ be the unital semigroup

$$\mathbb{F}_\theta^+ = \langle e_1, \dots, e_m, f_1, \dots, f_n : e_i f_j = f_{j'} e_{i'} \text{ when } \theta(i, j) = (i', j') \rangle.$$

Let $\mathcal{H}_\theta = \ell^2(\mathbb{F}_\theta^+)$ with orthonormal basis $\{\xi_w : w \in \mathbb{F}_\theta^+\}$. Define θ -commuting row-isometries $E = [E_1, \dots, E_m]$ and $F = [F_1, \dots, F_n]$ by

$$E_i \xi_w = \xi_{e_i w} \text{ and } F_j \xi_w = \xi_{f_j w}.$$

Let $\mathcal{A}_\theta = \overline{\text{alg}}^{\|\cdot\|} \{I, E_1, \dots, E_m, F_1, \dots, F_n\}$. We call this the *higher-rank noncommutative disc algebra*.

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Nonself-adjoint 2-graph algebras

We will be primarily interested in θ -commuting row-isometries (S, T) where both S and T are Cuntz-type. These are precisely the Cuntz-Krieger families for the 2-graph \mathbb{F}_θ^+ .

Definition

Let (S, T) be a pair of θ -commuting Cuntz-type row-isometries. We call the algebra

$$\mathcal{S} = \overline{\text{alg}}^{\text{WOT}} \{I, S_1, \dots, S_m, T_1, \dots, T_n\}$$

a *nonself-adjoint 2-graph algebra*.

Definition

Let S be a row-isometry. We call the algebra

$$\mathcal{S} = \overline{\text{alg}}^{\text{WOT}} \{I, S_1, \dots, S_m\}$$

a *free semigroup algebra*.

Theorem (Davidson, Katsoulis & Pitts (2001))

Let S be a row-isometry on \mathcal{H} . Let \mathcal{S} be the unital weakly closed algebra generated by S and let \mathcal{M} be the von-Neumann algebra generated by S .

Then there is a projection P in \mathcal{S} so that

- ① $P^\perp \mathcal{H}$ is an invariant subspace for S ,
- ② $\mathcal{S} = \mathcal{M}P + P^\perp \mathcal{S}P^\perp$,
- ③ $P^\perp \mathcal{S}P^\perp$ is “like” \mathfrak{L}_n .

The Structure of nonself-adjoint 2-graphs

Theorem (F. & Yang (2013))

Let (S, T) be Cuntz-type θ -commuting row-isometries on \mathcal{H} . Let \mathcal{S} be the nonself-adjoint 2-graph generated by S and T and let \mathcal{M} be the von-Neumann algebra generated by S and T .

Then there is a projection P in \mathcal{S} so that

- 1 $P^\perp \mathcal{H}$ is an invariant subspace for \mathcal{S} ,*
- 2 $\mathcal{S} = \mathcal{M}P + P^\perp \mathcal{S}P^\perp$.*

The Structure projection

Let (S, T) be a Cuntz-type representation of \mathbb{F}_θ^+ and let \mathcal{S} be the nonself-adjoint 2-graph algebra generated by (S, T) .
Note that $[S_1 T_1, S_1 T_2, \dots, S_m T_n]$ is a row-isometry.

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For any $k, l \geq 0$ we have a row-isometry

$$[ST]_{k,l} := [S_w T_u : |w| = k, |u| = l].$$

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For any $k, l \geq 0$ we have a row-isometry

$$[ST]_{k,l} := [S_w T_u : |w| = k, |u| = l].$$

Each of these row-isometries generates a free semigroup algebra inside \mathcal{S} . Let $\mathcal{S}_{k,l}$ be the free semigroup algebra generated by $[ST]_{k,l}$.

By Davidson-Katsoulis-Pitts each $\mathcal{S}_{k,l}$ has a structure projection $P_{k,l}$. Then

$$P = \bigwedge_{k,l > 0} P_{k,l}.$$

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Question

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Answer

Our setting is too general.

Example

Let S be any Cuntz-type row-isometry and let $T = S$. Then (S, T) are θ -commuting row-isometries (for some θ). So the nonself-adjoint 2-graph generated by (S, T) is just the free semigroup algebra generated by S .

The above example is a representation of a periodic 2-graph.

Periodicity of 2-graphs is a technical condition about the existence of repetition in infinite red-blue paths. If (S, T) is a Cuntz-type representation of an aperiodic 2-graph then there will necessarily be a strong relation between S and T making them behave more like a 1-graph than a 2-graph.

Lemma (Davidson & Yang (2009))

Let (S, T) be θ -commuting Cuntz-type row-isometries where \mathbb{F}_θ^+ is a periodic 2-graph. Then there are $a, b > 0$ such that $m^a = n^b$ such that

$$[S_v : |v| = a] = [T_u W : |u| = b],$$

where W is a unitary in the center of the C^ -algebra generated by S and T .*

The Structure of nonself-adjoint 2-graphs

Theorem (F. & Yang (2013))

Let (S, T) be Cuntz-type θ -commuting row-isometries on \mathcal{H} . Let \mathcal{S} be the nonself-adjoint 2-graph generated by S and T and let \mathcal{M} be the von-Neumann algebra generated by S and T

Then there is a projection P in \mathcal{S} so that

- 1 $P^\perp \mathcal{H}$ is an invariant subspace for \mathcal{S} ,
- 2 $\mathcal{S} = \mathcal{M}P + P^\perp \mathcal{S}P^\perp$.

Further, if θ defines an aperiodic 2-graph then there is a projection Q such that $Q \leq P^\perp$ and

- 3 $Q\mathcal{H}$ is an invariant subspace for \mathcal{S} ,
- 4 QSQ is "like" \mathfrak{L}_θ .

Theorem (Popescu)

Let $S = [S_1, \dots, S_n]$ be any row-isometry. Then

$$\mathcal{A} = \overline{\text{alg}}^{\|\cdot\|} \{I, S_1, S_2, \dots, S_n\}$$

is completely isometrically isomorphic to the noncommutative disc algebra \mathcal{A}_n .

This does not hold for isometric representations of 2-graphs. Not even for aperiodic 2-graphs:

Example

Let $L = [L_1, \dots, L_n]$ be the left regular representation of \mathbb{F}_n^+ and let $R = [R_1, \dots, R_n]$ be the right regular representation. Then $L_i R_j = R_j L_i$. It can be shown that $\overline{\text{alg}}^{\|\cdot\|} \{I, L_i, R_j\}$ is not completely isometrically isomorphic to \mathcal{A}_{id} .

However, in our setting something similar to Popescu's result does hold:

Theorem (F. & Yang 2013)

Let (S, T) be an isometric representation of an aperiodic 2-graph \mathbb{F}_θ^+ on a Hilbert space \mathcal{H} . Let

$$\mathcal{A} = \overline{\text{alg}}^{\|\cdot\|} \{I, S_1, \dots, S_m, T_1, \dots, T_n\}.$$

Suppose there is a Cuntz-type representation (S', T') of \mathbb{F}_θ^+ on a Hilbert space \mathcal{K} containing \mathcal{H} such that (S, T) is the restriction of (S', T') , i.e. each $S_i = S'_i|_{\mathcal{H}}$ and $T_j = T'_j|_{\mathcal{H}}$.

Then \mathcal{A} is completely isometrically isomorphic to \mathcal{A}_θ .