

CONTINUOUS  
SHEPHERD  
FAMILIES

joint work with

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Erdélyi 1941:

Orthogonal systems  $\begin{cases} \text{discrete } P_n, n \in \mathbb{N} \\ \text{continuous } P_\nu, \nu \in \Omega \subseteq \mathbb{C} \end{cases}$

$\equiv$  Eigenfunctions  $\begin{cases} \text{Same DIFF. EQ.,} \\ \text{different intervals.} \end{cases}$

• Example HERMITE

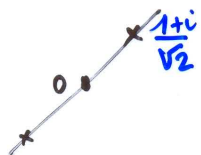
$$y''(t) - 2ty'(t) + 2\lambda y(t) = 0$$

$$\iff \left( \frac{d^2}{dt^2} - 2t \frac{d}{dt} \right) y = \lambda y$$

$\lambda = 2\lambda$

• On  $(-\infty, \infty)$ ,  $\begin{cases} \lambda = -2n, n \in \mathbb{N}. \\ y_n(t) = H_n(t) := (-1)^n e^{t^2} \frac{d^n}{dt^n} (e^{-t^2})(t). \end{cases}$

• On  $(-\infty\sqrt{i}, \infty\sqrt{i})$ ,  $\begin{cases} \lambda = -\frac{1}{2} + iy, y \in \mathbb{R}. \\ y_\gamma(t) = H_{-\frac{1}{2}+iy}(t), \end{cases}$



$$H_{-\frac{1}{2}}(t) = \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty e^{-s^2} e^{-2ts} s^{\frac{1}{2}-1} ds$$

LINK: inverse Mellin transform

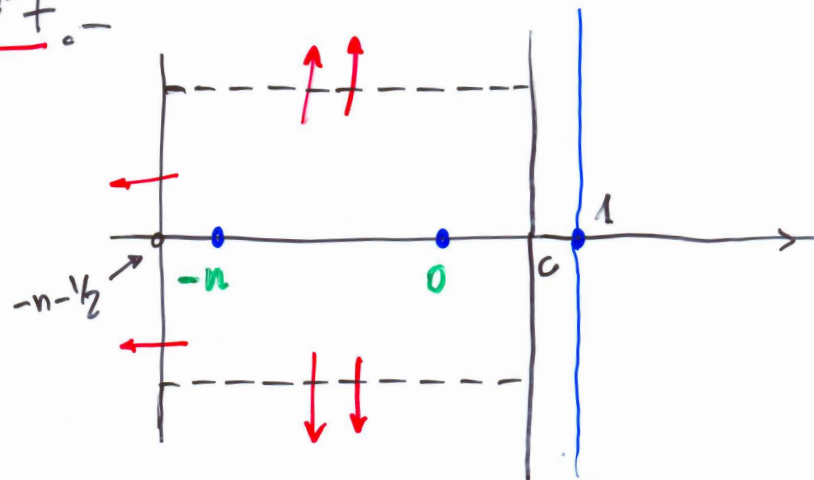
Lemma. -  $F \in \mathcal{H}(\{\operatorname{Re} z > -1\})$  s.t.  
 $|F(z)| \leq M e^{k|z|}, \operatorname{Re} z > 1 \ (k < \frac{\pi}{2})$



$$\frac{1}{2\pi i} \int_{\operatorname{Re} z = c > 0} \Gamma(z) F(-z) x^{-z} dz = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} F(n) x^n \quad (0 < x < 1)$$

(x > 0)

Pf. -



$$\Gamma(-n-\frac{1}{2}+iy) = \frac{(-1)^n \Gamma(\frac{1}{2}+iy)}{(1+\frac{1}{2}-iy)(n-\frac{1}{2}-iy) \dots (\frac{1}{2}+iy)} \quad \square$$

## Examples

(i)  $F(n) = H_n(t)$  "

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} H_n(t) x^n = \sum_{n=0}^{\infty} e^t \frac{d^n}{dx^n} \left[ e^{-(x+t)^2} \right]_{x=0} \frac{x^n}{n!}$$

$$= e^t e^{-(x+t)^2} = e^{-x^2} e^{-2xt}$$

$$= \frac{1}{2\pi i} \int_{\text{Re } z=c} \Gamma(z) H_{-z}(t) x^{-z} dz$$

or  $\Gamma(z) H_{-z}(t) = \int_0^{\infty} x^{z-1} e^{-x^2} e^{-2tx} dx =: \mathcal{M}[e^{-x^2-2tx}](z)$

$\text{Re } z > 0$

(ii)  $F(n) = t^n$  "

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n x^n = e^{-tx} = \int_{\text{Re } z=c} \Gamma(z) t^{-z} x^{-z} \frac{dz}{2\pi i};$$

that is,

$$t^{-z} \Gamma(z) = \int_0^{\infty} x^{z-1} e^{-tx} dx \quad [x = y/t]$$

## BINOMIAL TYPE POLYNOMIALS $F(n, t)$ :

$$\forall s, t > 0 \quad p_n(s+t) = \sum_{k=0}^n \binom{n}{k} p_k(s) p_{n-k}(t)$$

or  $q_n(s+t) = \sum_{k=0}^n q_k(s) q_{n-k}(t) \quad \text{"} \quad q_n := \frac{p_n}{n!}$

• Sheffer:  $\{\sigma_n\}$  "  $\sigma_n \longleftrightarrow q_n$  s.t

$$\sigma_n(s+t) = \sum_{k=0}^n q_k(s) \sigma_{n-k}(t)$$

• Put  $q^t(n) := q_n(t)$ , so

$$q^{s+t}(n) = \sum_{k=0}^n q^s(k) q^t(n-k)$$

$$\equiv q^{s+t} = q^s * q^t;$$

$$\sigma_t(n) := \sigma_n(t) \xrightarrow{\text{green}} \sigma_{s+t} = q^s * \sigma_t$$

• Umbral calculus = Study semigroups  
(or Sheffer families) in  $\mathbb{C}[x]$ ,  $\mathbb{C}[[x]]$   
for convolution (& duality methods).

Aim: Setting (algebra  $\mathcal{U}$ )

for  $\Gamma(z)F(-z; t)$  as above,,

special functions as  $\begin{cases} \text{semigroups} \\ \text{Sheffer} \end{cases}$  in  $\mathcal{U}$

• How? Simple:

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} F(n, t) x^n = e^{-t\psi(x)} = \int \Gamma(z) F(-z, t) x^{-z} dz$$

$$\text{so } F(-z, t) = \int_0^{\infty} x^{z-1} e^{-t\psi(x)} dx \quad \text{Re } z = c$$

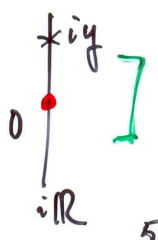
• Canonical:

$$\psi(x) = x > 0; \quad e^{-tx} = \int x^{-z} t^{-z} \Gamma(z) dz$$

so that  $\text{Re } z = c > 0$

$$\gamma^t: z \mapsto t^{-z} \Gamma(z)$$

must be semigroup in  $\mathcal{U}$

[  $\gamma^t$  not integrable at 0:  ]

(pre-)Umbral algebra  $\mathcal{U}$ :

$$\mathbb{C}^+ := \{z \mid \text{Re } z > 0\}$$

•  $\mathcal{U}$ :  $F \in \text{Hol}(\mathbb{C}^+)$  s.t.

$$\|F\|_{a,b} := \sup_{a \leq x \leq b} \int_{-\infty}^{\infty} |F(x+iy)| dy < \infty$$

with

$$\forall 0 < a \leq b$$

$$F * G(z) := \int_{\text{Re } w = c} F(z-w) G(w) dw \quad (0 < c < \text{Re } z)$$

•  $(\mathcal{U}, \|\cdot\|_{a,b})$  is

(i) LOCALLY CONVEX ALGEBRA (G.R. Allan)

$$\|F * G\|_{a,b} \leq \|F\|_{a-\varepsilon, b-\varepsilon} \|G\|_{\varepsilon, \varepsilon} \quad (0 < \varepsilon < a)$$

(ii) METRIZABLE ( $a = \frac{1}{n}; b = n \in \mathbb{N}$ )

(iii)  $\mathcal{U} \hookrightarrow \text{Hol}(\mathbb{C}^+)$  linearly

(iv)  $\mathcal{U}$  COMPLETE

THEOREM. - All continuous characters  $\varphi: \mathcal{U} \rightarrow \mathbb{C}$ ,  $\varphi \neq 0$ , are of the form

$\varphi = \varphi_\lambda$ ,  $\lambda > 0$ , where

$$[\widehat{F}(\lambda) =] \varphi_\lambda(F) := \frac{1}{2\pi i} \int_{\operatorname{Re} z = c} \lambda^{-z} F(z) dz \quad (F \in \mathcal{U}) \quad (c > 0)$$

•  $C_0(\mathbb{R}^+; \lambda^c)$ :

$f: (0, \infty) \rightarrow \mathbb{C}$  continuous  $\Rightarrow$

$$\lim_{\lambda \rightarrow 0, \infty} |f(\lambda) \lambda^c| = 0; \quad \|f\|_{\lambda^c} := \sup_{\lambda > 0} |f(\lambda) \lambda^c| \quad (\forall c > 0)$$

• Gelfand transform:

$$g: \mathcal{U} \rightarrow C_0(\mathbb{R}^+; \lambda^c)$$

$$F \mapsto \widehat{F}$$

$$g(F * G) = g(F) \cdot g(G), \quad F^t \mapsto e^{-t} \varphi$$

## SEMIGRUPOS in $\mathcal{U}$

For  $t, \alpha > 0$ ,

$$\gamma_\alpha^t(z) := \frac{1}{\alpha} t^{-z/\alpha} \Gamma(z/\alpha), \quad z \in \mathbb{C}^+.$$

•  $(\gamma_\alpha^t)_{t>0}$  is semigroup in  $\mathcal{U}$   
s.t.  $\widehat{\gamma_\alpha^t}(\lambda) = e^{-t\lambda^\alpha} \quad (\lambda > 0).$

$$\text{Pf.} - \int_0^\infty \lambda^{z-1} e^{-t\lambda^\alpha} d\lambda = [\lambda^z = r] = \frac{1}{\alpha} t^{-z/\alpha} \Gamma(z/\alpha) \quad \blacksquare$$

$$\text{So,} \quad (t+s)^{-z/\alpha} \Gamma(z/\alpha) = \frac{t^{-z/\alpha}}{2\pi di} \int \left(\frac{t}{s}\right)^{w/\alpha} \Gamma\left(\frac{z-w}{\alpha}\right) \Gamma\left(\frac{w}{\alpha}\right) dw \quad (s, t, \alpha > 0; z \in \mathbb{C}^+)$$

• Known:

$$(1-\lambda)_+^{t-1} = \int_{\operatorname{Re} z = c} x^{-z} B(t, z) \frac{dz}{2\pi i} \quad \therefore$$

$(\beta^t)_{t>0}$  is semigroup in  $\mathcal{U} \Rightarrow \widehat{\beta^t}(\lambda) = (1-\lambda)_+^t, \lambda > 0$   
where  $\beta^t(z) := B(z, t+1), z \in \mathbb{C}^+, t > 0.$

• Others

## APPLICATION to $L^1(\mathbb{R})$

$\forall s, t > 0, y \in \mathbb{R},$

$$\gamma_{\alpha}^{s,t}(y) := \frac{1}{2\pi\alpha} t^{-s/\alpha} t^{-iy/\alpha} \Gamma\left(\frac{s+iy}{\alpha}\right).$$

For  $\xi \in \mathbb{R},$

$$\begin{aligned} \mathcal{F}(\gamma_{\alpha}^{s,t})(\xi) &= \frac{e^{s\xi}}{2\pi i \alpha} \int_{\text{Re } z = s} (e^{\xi})^{-z} t^{-z/\alpha} \Gamma\left(\frac{z}{\alpha}\right) dz \\ &= e^{s\xi - t \cdot \exp(e^{\alpha\xi})} \end{aligned}$$

$\therefore (\gamma_{\alpha}^{s,t})_{s,t>0}$  is a bi-parameter semigroup in  $L^1(\mathbb{R})$

• (s=t). Put  $\Gamma_{\alpha}^t := \gamma_{\alpha}^{t,t}, t > 0.$

Then  $(\Gamma_{\alpha}^t)_{t>0}$  semigroup in  $L^1(\mathbb{R})$

s.t.  $\mathcal{F}(\Gamma_{\alpha}^t)(\xi) = e^{-t(e^{\alpha\xi} - \xi)}$   
 $\forall \xi \in \mathbb{R}.$

## SHEFFER & APPELL families in $\mathcal{U}$

DEF.  $(S_t)_{t>0} \subseteq \mathcal{U}$  is Sheffer if  $\exists (F^t)_{t>0}$  is semigroup in  $\mathcal{U} \Rightarrow$

$$S_{s+t} = F^s * S_t \quad (s, t > 0)$$

• For  $F^t = \gamma^t$ , Sheffer  $\equiv$  Appell

• PATTERN  $\sigma_t^* \equiv \{\sigma(n,t)\} \leftrightarrow \{q(n,t)\} \equiv q^t$

$$\widehat{\sigma}_{s+t}^*(x) = \widehat{\sigma_s^*} \cdot q^t(x) = \widehat{\sigma_s^*}(x) \cdot \widehat{q^t}(x)$$

$$\sum_{n=0}^{\infty} \frac{\sigma(n,s+t)}{n!} x^n = \sum_{n=0}^{\infty} \frac{\sigma(n,s)}{n!} x^n \cdot \sum_{n=0}^{\infty} \frac{q(n,t)}{n!} x^n$$

(1  $\rightarrow$  0<sup>+</sup>)

$$\left[ \sum_{n=0}^{\infty} \frac{\sigma(n,t)}{n!} x^n = \phi(x) \cdot e^{t\psi(x)} \right]$$

$$\Rightarrow \dots \rightsquigarrow \underline{S_t(z) := \int_0^{\infty} d^{z-1} \phi(-\lambda) e^{t\psi(-\lambda)} d\lambda}$$

HERMITE  $\sum_{n=0}^{\infty} H_n(t) \frac{x^n}{n!} = e^{-x^2} e^{2xt}$

We have

$$\frac{H_n(s+t)}{n! 2^n} = \sum_{k=0}^n \frac{H_k(s)}{k! 2^k} \frac{t^{n-k}}{(n-k)!}$$

Put  $h_t(z) := 2^z \Gamma(z) H_{-z}(t)$ ,  $z \in \mathbb{C}^+$

Recall,

$$H_{-z}(t) = \frac{1}{\Gamma(z)} \int_0^{\infty} x^{z-1} e^{-x^2} e^{-2tx} dx$$

$$= \frac{2^{-z}}{\Gamma(z)} \int_0^{\infty} \lambda^{z-1} e^{-\lambda^2/4} e^{-t\lambda} d\lambda$$

whence  $\widehat{h}_t(\lambda) = e^{-\lambda^2/4} e^{-t\lambda} \dots (h)_{t, t_0}$  Appell

• Remark. - In fact,

$$\widehat{h}_t(z) = \int_{-\infty}^{\infty} (t+is)^{-z} \Gamma(z) e^{-s^2} \frac{ds}{\sqrt{\pi}} = \int_{-\infty}^{\infty} \Gamma(z) e^{-s^2} ds$$

$$= \phi * \gamma^t(z), \text{ where}$$

$$\phi(z) = \int_{-\infty}^{\infty} \gamma^{is}(z) e^{-s^2} ds = \omega \frac{\pi^{1/2}}{2} \Gamma\left(\frac{1-z}{2}\right) \Gamma(z).$$

## LERCH

Apostol - Bernoulli polynomials  $\frac{x e^{tx}}{\alpha e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(t, \alpha)}{n!} x^n$   
 $|\alpha| \leq 1; t, \lambda > 0$   
 $(\alpha = 1, \text{Bernoulli})$

• Put

$$b_{t, \alpha}(z) := \int_0^{\infty} \lambda^{z-1} \frac{d e^{-t\lambda}}{1 - \alpha e^{-\lambda}} = \Gamma(z+1) \sum_{n=0}^{\infty} \frac{\alpha^n}{(n+t)^{z+1}}$$

$$= \Gamma(z+1) \cdot \underbrace{\Phi(\alpha, z+1, t)}_{\text{Lerch}} \in \mathcal{L}.$$

Then  $\widehat{b}_{t, \alpha}(\lambda) = \frac{\lambda}{1 - \alpha e^{-\lambda}} e^{-t\lambda}; \infty, \text{Appell}:$   
 $b_{t+s, \alpha} = b_{t, \alpha} * \gamma^t.$

• Similarly,

Apostol - Euler pol.:  $\frac{2e^{tx}}{\alpha e^x + 1} = \sum_{n=0}^{\infty} \frac{E_n(t, \alpha)}{n!} x^n$

$\leadsto e_{t, \alpha}(z) := \int_0^{\infty} \frac{2e^{-t\lambda}}{1 + \alpha e^{-\lambda}} \lambda^{z-1} d\lambda$   
 $= 2\Gamma(z) \sum_{n=0}^{\infty} \frac{(-1)^n}{(t+n)^z} \quad (\text{zetaeta-Lerch at } -1/2).$   
 $\alpha = 1$

## CHARLIER

For  $x, t > 0$ ,  $\gamma^{1+x}(t) = (\gamma^x * \gamma^1)(t)$

$$\equiv (1+x)^{-t} \Gamma(t) = \frac{1}{2\pi i} \int_{\text{Re } z=c < t} x^{-z} \Gamma(z) \Gamma'(t-z) dz$$

or

$$(1+x)^{-t} = \frac{1}{2\pi i} \int_{\text{Re } z=c} x^{-z} B(z, t-z) dz, \quad \left[ \begin{array}{l} B(z, t-z) \\ 0 < \text{Re } z < t \end{array} \right]$$

•  $\forall s, t > 0$ ,  $(\gamma B)^{s,t}(z) := \frac{1^{-z}}{2\pi i} \int_{\text{Re } w=c} s^w \Gamma(z-w) B(w, t-w) dw$   
 $\in \mathcal{L}$

with  $\widehat{(\gamma B)^{s,t}}(\lambda) = e^{-s\lambda} (1+\lambda)^{-t}$ ,  $\lambda > 0$ .

• Charlier pol.:  $e^x (1-s^{-1}x)^{-t} = \sum_{n=0}^{\infty} \frac{C_n(-t, s)}{n!} x^n$

so

$$C^{t,s}(z) := \int_0^{\infty} y^{z-1} e^{-y} (1+s^{-1}y)^{-t} dy$$

$$= s^z \int_0^{\infty} \lambda^{z-1} e^{-s\lambda} (1+\lambda)^{-t} d\lambda$$

$$= s^z (\gamma B)^{s,t}(z).$$

## ABEL $a > 0$ , $g(y) := y e^{ay}$ ( $y \geq 0$ ).

|| pol.:

$$e^t g^{-1}(x) = \sum_{n=0}^{\infty} \frac{a_n(t)}{n!} x^n, \quad a_n(t) = t(t-an)^{n-1}$$

•  $\Gamma(z) A_t(z) := \int_0^{\infty} x^{z-1} e^{-t} g^{-1}(x) dx$

$$= \int_0^{\infty} (1+au) u^{z-1} e^{-t(t-au)u} du$$

$$= \frac{\Gamma(z)}{(t-az)^z} + \frac{a \Gamma(z+1)}{(t-az)^{z+1}} = \frac{t \Gamma(z)}{(t-az)^{z+1}}$$

$\neq 0 < \text{Re } z < \frac{t}{a}$

DEF.

$$(\gamma A)^{s,t}(z) := \frac{t \Gamma(z)}{2\pi i} \int_{\text{Re } w=c} s^{z-w} B(w, z-w) \frac{dw}{(t-aw)^{w+1}}$$

$\in \mathcal{L}$  ;

$$\widehat{(\gamma A)^{s,t}}(\lambda) = e^{-s\lambda} e^{-t} g^{-1}(\lambda) \quad ; \lambda > 0$$

Gegenbauer, Laguerre, Jacobi, ...