# Weak amenability of the Fourier algebra

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# Fourier-Stieltjes algebra

$G$: LCG, $\mathcal{H}$: Hilbert space, $\pi : G \to \mathcal{U}(\mathcal{H})$: unitary rep.

$\xi *_\pi \eta(x) := \langle \pi(x)\xi, \eta \rangle$ coeff func associated with $\pi$.

$\Sigma = \{$equiv classes of unitary rep.$\}$

## Definition

- $\mathrm{B}(G) := \{\xi *_\pi \eta : \ \pi \in \Sigma, \xi, \eta \in \mathcal{H}_\pi\}$.
- For $u \in \mathrm{B}(G)$, $\|u\|_{\mathrm{B}(G)} = \inf\{\|\xi\|_{\mathcal{H}_\pi} \|\eta\|_{\mathcal{H}_\pi} : \ u = \xi *_\pi \eta\}$.
- $(\mathrm{B}(G), \ +, \ \cdot, \ \| \cdot \|_{\mathrm{B}(G)})$ Ban alg.

## Example

- $G$: Abelian, $\mathrm{B}(G) = \mathcal{F}(M(\hat{G}))$, with $\|\mathcal{F}\mu\|_A = \|\mu\|$.

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- $\mathrm{A}_\pi(G)$ consists of $u = \sum_{n=1}^\infty \xi_n *_\pi \eta_n$, $\sum_{i=1}^\infty \|\xi_i\|\|\eta_i\| < \infty$.
- $\|u\|_{\mathrm{B}(G)} = \inf\{\sum_{n=1}^\infty \|\xi_n\|\|\eta_n\| : \ u$ as above$\}$.
- $\mathrm{A}_\pi(G)$ closed subspace of $\mathrm{B}(G)$.

## Example (Fourier algebra)

$\mathrm{A}(G) = \mathrm{A}_\lambda(G)$.

**Note.** $\mathrm{A}(G) := \{\xi *_\lambda \eta : \ \xi, \eta \in L^2(G)\}$ ideal in $B(G)$.

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$\mathcal{A}$: Ban alg with character $\phi : \mathcal{A} \to \mathbb{C}$.

- $\mathbb{C}$ is an $\mathcal{A}$-bimodule via

$$a \cdot z = z \cdot a = \phi(a)z, \forall a \in \mathcal{A}, z \in \mathbb{C}.$$

- A cts linear $D : \mathcal{A} \to \mathbb{C}$ is a point derivation at $\phi$ if

$$D(ab) = \phi(a)D(b) + \phi(b)D(a).$$

**Example**

Character $\phi_x : \mathrm{A}(G) \to \mathbb{C},\ \phi_x(u) = u(x)$.
If $D : \mathrm{A}(G) \to \mathbb{C}$ pt derivation at $\phi_x$ then $D = 0$.

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## Example

- $\mathcal{A}$ Ban $\mathcal{A}$-bimodule (algebra multiplication=module action).
- $\mathcal{A}^*$ Ban $\mathcal{A}$-bimodule via $a \cdot f(x) = f(xa), \quad f \cdot a(x) = f(ax)$.

$D : \mathcal{A} \to X$ derivation if $D(ab) = D(a) \cdot b + a \cdot D(b)$.

## Definition. Bade-Curtis-Dales

comm Ban alg $\mathcal{A}$ is weakly amenable (w. amen) if every bndd der $D : \mathcal{A} \to \mathcal{A}^*$ is zero.

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## Some results

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For which groups $G$, $\mathrm{A}(G)$ is w. amen?

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For which groups $G$, $\mathrm{A}(G)$ is w. amen?

## Known results

- [Johnson, 91] $L^1(G)$ is always w. amen.
- [Johnson, 92] $\mathrm{A}(\mathrm{SO}_3(\mathbb{R}))$ not w. amen.
- [Johnson, Forrest, 90s] $G$ totally disconnected $\Rightarrow \mathrm{A}(G)$ w. amen.

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## Observation

$\mathcal{A}, \mathcal{B}$ comm Ban alg $\phi : \mathcal{A} \to \mathcal{B}$ cts hom with dense range.

- If $D : \mathcal{B} \to \mathcal{B}^*$ cts nonzero der then $\phi^* \circ D \circ \phi : \mathcal{A} \to \mathcal{A}^*$ cts nonzero der.
- $\mathcal{B}$ not w. amen $\Rightarrow$ $\mathcal{A}$ not w. amen.

## Proposition

$G$ LCG, $H$ closed subgroup.

- Restriction map $r : A(G) \to A(H)$ is cts onto hom.
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# Evidence to support the conjecture

$G_e$ connected component of identity.
**Conj.** $\mathrm{A}(G)$ w. amen $\Leftrightarrow$ $G_e$ abelian.

## Known results

- [Forrest-Runde, 04] $G_e$ abelian $\Rightarrow$ $\mathrm{A}(G)$ w. amen.
- [Johnson, 92] $\mathrm{A}(\mathrm{SO}_3(\mathbb{R}))$ not w. amen.
- [Plymen, 01] $G$ cpt Lie gp and $\mathrm{A}(G)$ w. amen $\Rightarrow$ $G_e$ abelian.
- [Forrest-Samei-Spronk, 09] $G$ cpt gp and $\mathrm{A}(G)$ w. amen $\Rightarrow$ $G_e$ abelian.

## Theorem (Choi-G., 13)

- *$\mathrm{A}(ax+b)$ is not w. amen.*
- *$G$ non-cpt connected semisimple Lie gp $\Rightarrow$ $\mathrm{A}(G)$ is not w. amen.*

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**Thm.** $\mathrm{A}(\mathrm{SO}_3(\mathbb{R}))$ not w. amen [Johnson, 92].

**Proof.**

$\mathrm{A}(\mathrm{SO}(3)) \to \mathrm{A}(\mathrm{SU}(2))$ isometric inclusion.

Find $D : \mathrm{A}(\mathrm{SU}(2)) \to \mathrm{A}(\mathrm{SU}(2))^*$ s.t. $D|_{\mathrm{A}(\mathrm{SO}(3))} \neq 0$.

For $\phi \in \mathbb{R}$, $s_\phi = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}$.

$\partial_\phi : C^1(\mathrm{SU}(2)) \to C(\mathrm{SU}(2))$, $\partial_\phi f(p) = \frac{\partial}{\partial\phi} f(ps_\phi)\Big|_{\phi=0}$.

$D_\flat : C^1(\mathrm{SU}(2)) \to \mathrm{A}(\mathrm{SU}(2))^*$, $D_\flat(f)(g) = \int_{\mathrm{SU}(2)} (\partial_\phi f) g \, d\mu$.

$|D_\flat(f)(g)| \leq C\|f\|_{\mathrm{A}}\|g\|_{\mathrm{A}}$.

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**Note.** Johnson used $\mathrm{A}_\gamma(G)$ in his proof.

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# An important non-compact group

## Definition

- $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : \ a \in \mathbb{R}_+^*, b \in \mathbb{R} \right\}.$
- $\mathbb{R}_+^* := (\mathbb{R}_+, *)$ with Haar measure $t^{-1}dt$.
  $\mathbb{R}_+^*$ acts on $\mathbb{R}$ by mult.
- $G \simeq \mathbb{R} \rtimes \mathbb{R}_+^* = \{(b, a) : b \in \mathbb{R}, a \in \mathbb{R}_+^*\}.$

## Properties of $G = ax + b$

Left Haar measure: $d\mu(a, b) = a^{-2} da\, db.$

Modular function: $\Delta(a, b) = \frac{1}{a}.$

# Representation theory of $ax+b$

## $\infty$-dim irred rep

Mackey machine for induced rep:

$$\pi_\pm : G \to \mathcal{U}(L^2(\mathbb{R}_+^*, dt/t)),$$

- $\pi_+(b,a)\xi(t) = e^{-2\pi ibt}\xi(at)$,
- $\pi_-(b,a)\xi(t) = e^{+2\pi ibt}\xi(at)$.

## coefficient functions

- $(\xi *_{\pi_+} \eta)(b,a) = \int_0^\infty e^{-2\pi ibt}\xi(at)\overline{\eta(t)}\, t^{-1}dt$,
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$\mathcal{H}= L^2(\mathbb{R}_+^*, dt/t)$.
$\mathcal{F}$: Fourier transform on $\mathbb{R}$.

- $K : \mathcal{H} \to \mathcal{H}, K\xi(t) = t\xi(t)$.
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**Definition**

Irred rep $\pi$ is square integrable if $\exists\, 0 \neq \xi *_\pi \eta \in L^2(G)$.

**Note.** $\pi$ sq. integ. iff $\pi < \lambda$.

**Orthogonality rel in $ax + b$**

$\eta_1, \eta_2, \xi_1, \xi_2 \in C_c^2(\mathbb{R}_+^*)$. Then

$\langle \xi_1 *_{\pi_+} \eta_1, \xi_2 *_{\pi_+} \eta_2 \rangle_{L^2(G)} = \langle \eta_2, \eta_1 \rangle_{\mathcal{H}} \langle K^{-\frac{1}{2}} \xi_1, K^{-\frac{1}{2}} \xi_2 \rangle_{\mathcal{H}}.$

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**Corollary**

$\pi_+$ and $\pi_-$ are sq. integ.

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## Corollary

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## Norm on $\mathrm{A}_{\pi_\pm}$

$\|\xi *_{\pi_\pm} \eta\|_{\mathrm{A}(G)} = \|\xi\|\|\eta\|.$

## Proposition

$\mathrm{A}(G) = \mathrm{A}_{\pi_+} \oplus_1 \mathrm{A}_{\pi_-}$.

## Proof.

- $\{\xi *_{\pi_\pm} \eta : \xi, \eta \in C_c(\mathbb{R}_+)\}$ is dense in $L^2(G)$.
- $\langle \lambda(x)(\xi *_{\pi_\pm} \eta), \xi' *_{\pi_\pm} \eta' \rangle = \alpha \langle \pi_\mp(x)\overline{\eta}, \overline{\eta'} \rangle$.
- $\langle \lambda(x)(\xi *_{\pi_\pm} \eta), \xi' *_{\pi_\mp} \eta' \rangle = 0$.

$\square$

# Defining the derivation

- $\mathcal{C} = \mathrm{Span}\{\xi *_{\pi_\pm} \eta : \xi, \eta \in C_c^\infty(\mathbb{R}_+^*)\}$.
- $D_1 : \mathcal{C} \to \mathcal{C}$, $D_1(f)(b, a) = \frac{-a}{2\pi i}\frac{\partial}{\partial b} f(b, a)$.
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## Definition

$\mathbb{H}_r := \{(p, q, e^{2\pi i\theta}) : p, q \in \mathbb{R}, \theta \in [0, 1)\}$.
$(p, q, e^{2\pi i\theta}) \cdot (p', q', e^{2\pi i\theta'}) = (p + p', q + q', e^{2\pi i(\theta+\theta')} e^{\pi i(pq' - qp')})$.

## Representations

For $n \in \mathbb{Z} \setminus \{0\}$, $\mathrm{Sch}_n : \mathbb{H}_r \to \mathcal{U}(L^2(\mathbb{R}))$ is defined by

$$\mathrm{Sch}_n(p, q, e^{2\pi i\theta})\xi(x) = e^{2\pi inq(-x+\frac{p}{2})} e^{2\pi in\theta} \xi(-p + x).$$

## Proposition

- $\langle \xi_1 *_{\pi_n} \eta_1, \xi_2 *_{\pi_n} \eta_2 \rangle_{L^2(G)} = \frac{1}{n} \langle \eta_2, \eta_1 \rangle_{\mathcal{H}} \langle \xi_1, \xi_2 \rangle_{\mathcal{H}}$.
- $\langle \xi_1 *_{\pi_n} \eta_1, \xi_2 *_{\pi_m} \eta_2 \rangle_{L^2(G)} = 0$, whenever $n \neq m$.
- $\mathrm{A}(\mathbb{H}_r) = \oplus_{n \neq 0} \mathrm{A}_{\mathrm{Sch}}(\mathbb{H}_r) \oplus \mathrm{A}(\mathbb{H}_r : \mathbb{T})$.

**Question**

Let $G = \mathbb{R}^2 \rtimes \mathrm{So}(2)$. Is $\mathrm{A}(G)$ weakly amenable?