Finite summability in noncommutative geometry

Magnus Goffeng
joint work with Bram Mesland

Institut für Analysis
Leibniz Universität Hannover

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Banach Algebras and Applications

Historical introduction

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1. Given a C^*-algebra A, can any $x \in K^*(A)$ be represented by a finitely summable Fredholm module?
2. If not, can one determine which $x \in K^*(A)$ can be represented in this way?

Some results in the negative direction

Rave 2012

1. If $x \in K^0(A)$ can be represented by a K-cycle that is finitely summable on all of A, x can be represented by $(\pi, \mathcal{H}, 0)$ where $\mathcal{H}$ is finite-dimensional. If $x \in K^1(A)$ can be represented by a K-cycle that is finitely summable on all of A, $x = 0$.
2. If Γ is a discrete group, $A = C^*(\Gamma)$ and $x \in K^*(A)$ can be represented by a K-cycle that is finitely summable on $\ell^1(\Gamma)$, the same statement as above holds true.

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Puschnigg 2008

If Γ is a higher rank lattice (cofinite discrete subgroup of product of higher rank Lie group), any $x \in K^0(C_r^*(\Gamma))$ represented by a Fredholm module that is finitely summable on $\mathbb{C}[\Gamma]$, satisfies $x = 0$.

The nail in the coffin...

A negative answer to the first question

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If $\mathcal{A} \subseteq \oplus_{n=1}^{\infty} C(S^{2n-1})$ is dense and holomorphically closed, $\oplus_n^{alg} C(S^{2n-1}) \subseteq \mathcal{A}$.

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Bounded transform: analytic K-cycles from spectral triples

The triple $(\mathrm{id}, \mathcal{H}, F_D)$ is an analytic K-cycle for $A := \overline{\mathcal{A}}^{\mathcal{B}(\mathcal{H})}$, where

$$F_D := D(1 + D^2)^{-1/2} \quad \text{or} \quad F_D = D|D|^{-1} \quad \textit{if } D \text{ is invertible.}$$

This operation sends p-summable spectral triples to p-summable Fredholm modules. Any Fredholm module can be realized as the bounded transform of a spectral triple.

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Emerson-Nica 2012

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For an $A \in M_n(\{0,1\})$ such that no column nor row is 0, the Cuntz-Krieger algebra O_A is the universal C^*-algebra generated by partial isometries $S_1, \dots, S_n$ such that the projections $P_i := S_i S_i^*$ are orthogonal and

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Under some rather mild conditions on A, O_A is simple and purely infinite.

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The Cuntz algebra

If $A_{ij} = 1$ for all i, j, one writes $O_n := O_A$– the Cuntz algebra. It holds that

$$K_0(O_n) \cong K^1(O_n) \cong \mathbb{Z}/(n-1) \quad \text{and} \quad K_1(O_n) \cong K^0(O_n) \cong \mathbb{Z}/(n-1).$$

Finite summability in Cuntz-Krieger algebras

G.-Mesland 2012

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The proof of this result is based on a duality result of Kaminker-Putnam:

$$K_*(O_A) \cong K^{*+1}(O_{A^T})$$

which is induced from a K-homology element constructed from an extension

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What clinches the deal is that this short exact sequence admits a completely positive splitting that is multiplicative up to p-summable errors.