Convolution algebras associated with locally compact quantum groups

Zhiguo Hu

University of Windsor

Joint work with Matthias Neufang and Zhong-Jin Ruan

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Von Neumann algebraic LCQGs

Let $\mathbb{G} = \{\mathcal{M}, \Gamma, \varphi, \psi\}$ be a von Neumann algebraic locally compact quantum group (Kustermans-Vaes 03)

That is

- $\mathcal{M}$ is a von Neumann algebra
- $\Gamma : \mathcal{M} \longrightarrow \mathcal{M}\bar{\otimes}\mathcal{M}$ is a co-multiplication; i.e., Γ is a normal and unital $*$-homomorphism satisfying $(\Gamma \otimes id)\Gamma = (id \otimes \Gamma)\Gamma$
- φ and ψ are respectively normal semifinite faithful left and right invariant weights on $(\mathcal{M}, \Gamma)$

Example. Let G be a locally compact group. Then

$$\mathbb{G} = (L_\infty(G), \Gamma, \varphi, \psi)$$

is a commutative LCQG, where the co-multiplication

$$\Gamma : L_\infty(G) \longrightarrow L_\infty(G)\bar{\otimes}L_\infty(G)$$

is given by $\Gamma(f)(s,t) = f(st)$, and φ and ψ are the left and right Haar integrals over G

In Fact, Γ is just the adjoint of the convolution on $L_1(G)$, and every commutative LCQG is of this form

Von Neumann algebraic LCQGs

For general $\mathbb{G} = \{\mathcal{M}, \Gamma, \varphi, \psi\}$

- $$L_\infty(\mathbb{G}) = \mathcal{M} \qquad L_1(\mathbb{G}) = \mathcal{M}_* \qquad L_2(\mathbb{G}) = H_\varphi$$

- $L_1(\mathbb{G})$ is a completely contractive Banach algebra under
$$\star = \Gamma_* : L_1(\mathbb{G})\widehat{\otimes}L_1(\mathbb{G}) \longrightarrow L_1(\mathbb{G})$$

- For $L_\infty(G)$, $\star$ is the convolution on $L_1(G)$

- For $VN(G)$ (the von Neumann algebra generated by the left regular repn of G), $\star$ is the pointwise multiplication on $A(G)$

Von Neumann algebraic LCQGs

- Every LCQG $\mathbb{G}$ has its *dual* LCQG $\widehat{\mathbb{G}}$
- The Pontryagin duality theorem holds: $\widehat{\widehat{\mathbb{G}}} \cong \mathbb{G}$
- As LCQGs, $L_\infty(G)$ and $VN(G)$ are dual of each other

The reduced C^*-algebraic LCQG $C_0(\mathbb{G})$

$C_0(\mathbb{G}) \; = \;$ the reduced quantum group C^*-algebra of $\mathbb{G}$

- $C_0(\mathbb{G})$ is a w^*-dense C^*-subalgebra of $L_\infty(\mathbb{G})$
- $\mathbb{G}$ is *compact* if $C_0(\mathbb{G})$ is unital

  $\mathbb{G}$ is *discrete* if $L_1(\mathbb{G})$ is unital ($\Longleftrightarrow$ $\widehat{\mathbb{G}}$ is compact)

The quantum measure algebra $M(\mathbb{G})$

- The co-multiplication $\Gamma : C_0(\mathbb{G}) \longrightarrow M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$ induces a completely contractive multiplication $\star$ on $C_0(\mathbb{G})^*$

$$M(\mathbb{G}) \; = \; (C_0(\mathbb{G})^*, \star)$$

- $\star$ is separately w^*-w^* continuous on $M(\mathbb{G})$
- $L_1(\mathbb{G})$ is canonically a closed ideal in $M(\mathbb{G})$ via $f \mapsto f|_{C_0(\mathbb{G})}$

The LUC-space associated with $L_1(\mathbb{G})$

$$LUC(\mathbb{G}) \;=\; \langle L_\infty(\mathbb{G}) \star L_1(\mathbb{G}) \rangle$$

- $C_0(\mathbb{G}) \;\subseteq\; LUC(\mathbb{G}) \;\subseteq\; M(C_0(\mathbb{G})) \;\subseteq\; L_\infty(\mathbb{G})$
- $LUC(\mathbb{G})^*$ is a Banach algebra in a canonical way
- $LUC(\mathbb{G})^* \;=\; M(\mathbb{G}) \oplus C_0(\mathbb{G})^\perp$

The convolution algebra $(T(L_2(\mathbb{G})), \triangleright)$

Let $V \in B(L_2(\mathbb{G}) \otimes L_2(\mathbb{G}))$ be the *right* multiplicative unitary of $\mathbb{G}$:

$$\Gamma(x) \;=\; V(x \otimes 1)V^* \quad (x \in L_\infty(\mathbb{G}))$$

Then V induces on $B(L_2(\mathbb{G}))$ the co-multiplication

$$\Gamma^r : B(L_2(\mathbb{G})) \longrightarrow B(L_2(\mathbb{G}))\bar{\otimes}B(L_2(\mathbb{G})),\; x \longmapsto V(x \otimes 1)V^*$$

and hence on $T(L_2(\mathbb{G}))$ the multiplication

$$\triangleright = (\Gamma^r)_* \;:\; T(L_2(\mathbb{G}))\widehat{\otimes}T(L_2(\mathbb{G})) \longrightarrow T(L_2(\mathbb{G}))$$

- $(T(L_2(\mathbb{G})), \triangleright)$ is a completely contractive Banach algebra
- Some properties of $\mathbb{G}$ can be characterized via $(T(L_2(\mathbb{G})), \triangleright)$

Regular q. group $\mathbb{G}$ and regular semigroup $(T(L_2(\mathbb{G})), \triangleright)$

- For the left multiplicative unitary W of $\mathbb{G}$ on $L_2(\mathbb{G}) \otimes L_2(\mathbb{G})$,

  $$C(W) := \{(\iota \otimes \omega)(\Sigma W) : \omega \in T(L_2(\mathbb{G}))\} \subseteq B(L_2(\mathbb{G}))$$

  where Σ is the flip map on $L_2(\mathbb{G}) \otimes L_2(\mathbb{G})$

- $\mathbb{G}$ is *regular* if $K(L_2(\mathbb{G})) = \langle C(W) \rangle$ (Baaj-Skandalis 93)

- All compact and discrete quantum groups are regular

  Kac algebras, in particular, $L_\infty(G)$ and $VN(G)$, are regular

- There exist non-semiregular $\mathbb{G}$ (Baaj-Skandalis-Vaes 03)

Recall: $T(L_2(\mathbb{G})) = K(L_2(\mathbb{G}))^*$. So, the *topological centres* of the semigroup $(T(L_2(\mathbb{G})), \triangleright)$ can be defined:

$$\mathfrak{Z}_t^{(\ell)}(T(L_2(\mathbb{G}))) = \{\gamma \in T(L_2(\mathbb{G})) : \omega \mapsto \gamma \triangleright \omega \text{ is } w^*\text{-cont}\}$$

$$\mathfrak{Z}_t^{(r)}(T(L_2(\mathbb{G}))) = \{\gamma \in T(L_2(\mathbb{G})) : \omega \mapsto \omega \triangleright \gamma \text{ is } w^*\text{-cont}\}$$

Theorem

1. $\mathbb{G}$ is regular $\Longleftrightarrow$ the semigroup $(T(L_2(\mathbb{G})), \triangleright)$ is right regular
2. $\mathbb{G}$ is discrete $\Longleftrightarrow$ the semigroup $(T(L_2(\mathbb{G})), \triangleright)$ is regular

The convolution algebras $(T(L_2(\mathbb{G})), \triangleright)$ and $L_1(\mathbb{G})$

- $(T(L_2(\mathbb{G})), \triangleright) \longrightarrow L_1(\mathbb{G}),\ \omega \longmapsto \omega|_{L_\infty(\mathbb{G})}$ is a surjective and completely contractive algebra homomorphism so that

$$(L_1(\mathbb{G}), \star) \quad \cong \quad (T(L_2(\mathbb{G})), \triangleright) \,/\, L_\infty(\mathbb{G})_\perp$$

- $(T(L_2(\mathbb{G})), \triangleright)$ is seen as the right lifting convolution algebra of $L_1(\mathbb{G})$ via the right fundamental unitary V of $\mathbb{G}$
- The left lifting convolution algebra $(T(L_2(\mathbb{G})), \triangleleft)$ is defined via the left fundamental unitary W of $\mathbb{G}$

Recall: $\mu \in RM(T(L_2(\mathbb{G})))$ if $\mu(\omega \triangleright \gamma) = \omega \triangleright \mu(\gamma)$

Then $\mu^* : B(L_2(\mathbb{G})) \longrightarrow B(L_2(\mathbb{G}))$, $\mu^*(L_\infty(\mathbb{G})) \subseteq L_\infty(\mathbb{G})$,

$$\Pi \; : \; RM(T(L_2(\mathbb{G}))) \longrightarrow RM(L_1(\mathbb{G})), \; \mu \longmapsto (\mu^*|_{L_\infty(\mathbb{G})})_*$$

and Π is injective.

Theorem

The map Π defines a completely isometric algebra isomorphism

$$RM_{cb}(T(L_2(\mathbb{G}))) \;\cong\; RM_{cb}(L_1(\mathbb{G}))$$

Combining with the representation theorem

$$RM_{cb}(L_1(\mathbb{G})) \;\cong\; CB^{\sigma,\, L_\infty(\mathbb{G})}_{L_\infty(\widehat{\mathbb{G}})}(B(L_2(\mathbb{G})))$$

by Junge-Neufang-Ruan (09) (Neufang-Ruan-Spronk (08) for the $L_\infty(G)$ and $VN(G)$ cases), we have

Corollary

The commutative diagram of comp isometric isomorphisms holds

$$\begin{array}{ccc} RM_{cb}(T(L_2(\mathbb{G}))) & \xrightarrow{\Pi} & RM_{cb}(L_1(\mathbb{G})) \\ \downarrow & & \downarrow \\ CB^{\sigma}_{T(L_2(\mathbb{G}))}(B(L_2(\mathbb{G}))) & = & CB^{\sigma, L_\infty(\mathbb{G})}_{L_\infty(\widehat{\mathbb{G}})}(B(L_2(\mathbb{G}))) \end{array}$$

Arens products on A^{**}

Let A be a Banach algebra. The *left Arens product* $\Box$ on A^{**} is naturally defined when A is considered as a *left* A-module:

for $a, b \in A$, $f \in A^*$, and $m, n \in A^{**}$, we have

$$\langle f \cdot a, b \rangle = \langle f, ab \rangle, \ \langle n \Box f, a \rangle = \langle n, f \cdot a \rangle, \ \langle m \Box n, f \rangle = \langle m, n \Box f \rangle$$

Similarly, the *right* Arens product $\diamondsuit$ on A^{**} is defined when A is considered as a right A-module

- Both $\Box$ and $\diamondsuit$ extend the multiplication on A
- A is said to be *Arens regular* if $\Box$ and $\diamondsuit$ coincide

Topological centres of A^{**}

- $(A^{**}, \Box)$ is a *right topological semigroup* under w^*-topology

  for any fixed $m \in A^{**}$, $n \longmapsto n\Box m$ is w^*-w^* continuous

  That is, the semigroup $(A^{**}, \Box)$ is right regular

- Similarly, $(A^{**}, \Diamond)$ is a *left topological semigroup*

- The topological centres of $(A^{**}, \Box)$ and $(A^{**}, \Diamond)$ are

$$\mathfrak{Z}_t(A^{**}, \Box) = \{m \in A^{**} : n \longmapsto m\Box n \text{ is } w^*\text{-}w^* \text{ cont}\}$$

$$\mathfrak{Z}_t(A^{**}, \Diamond) = \{m \in A^{**} : n \longmapsto n\Diamond m \text{ is } w^*\text{-}w^* \text{ cont}\}$$

  simply called the left and right topological centres of A^{**}

Topological centres, Arens regularity, and SAI

- $A \subseteq \mathfrak{Z}_t(A^{**}, \Box) \subseteq A^{**} \qquad A \subseteq \mathfrak{Z}_t(A^{**}, \Diamond) \subseteq A^{**}$
- $\mathfrak{Z}_t(A^{**}, \Box) = A^{**} \iff A \text{ is AR} \iff \mathfrak{Z}_t(A^{**}, \Diamond) = A^{**}$
- A is *left strongly Arens irreg* if $\mathfrak{Z}_t(A^{**}, \Box) = A$ (Dales-Lau 05)

  Similarly, A is said to be *right SAI* if $\mathfrak{Z}_t(A^{**}, \Diamond) = A$

  A is *SAI* if it is both left SAI and right SAI
- Every $L_1(G)$ is SAI (Lau-Losert 88)
- For many amenable G, $A(G)$ is SAI (Lau-Losert 93, 05, etc.)

SAI of the algebra $(T(L_2(\mathbb{G})), \triangleright)$

However, as a lifting of $L_1(\mathbb{G})$, the algebra $(T(L_2(\mathbb{G})), \triangleright)$ is not SAI unless $\mathbb{G}$ is finite. That is, we have

Theorem

T. F. A. E.

1. The convolution algebra $(T(L_2(\mathbb{G})), \triangleright)$ is SAI
2. $\mathbb{G}$ is finite (i.e., $\dim(L_\infty(\mathbb{G})) < \infty$)

Non-AR and non-SAI

Corollary

Let $\mathbb{G}$ be an infinite *co-amenable* locally compact quantum group.

If either (i) $\mathfrak{Z}_t(LUC(\mathbb{G})^*) = M(\mathbb{G})$

or (ii) $\mathbb{G}$ is *amenable*,

then

$$T(L_2(\mathbb{G})) \subsetneqq \mathfrak{Z}_t(T(L_2(\mathbb{G}))^{**}, \diamondsuit) \subsetneqq T(L_2(\mathbb{G}))^{**}$$

In particular, these inequalities hold if G is an infinite l. c. group and $L_\infty(\mathbb{G}) = L_\infty(G)$ or $L_\infty(\mathbb{G}) = VN(G)$ with G amenable.

The discrete quantum group case

Let $\mathbb{G}$ be discrete and e be a right identity of $(T(L_2(\mathbb{G})), \triangleright)$. Then

$$L_1(\mathbb{G}) \longrightarrow T(L_2(\mathbb{G})),\ f \longmapsto e \triangleright \omega$$

is an isometric homomorphism, where $\omega \in T(L_2(\mathbb{G})), \omega|_{L_\infty(\mathbb{G})} = f$

Theorem

For any infinite discrete quantum group $\mathbb{G}$ with $L_1(\mathbb{G})$ SAI, we have

$$T(L_2(\mathbb{G})) = L_1(\mathbb{G}) \oplus L_\infty(\mathbb{G})_\perp$$

$$\mathfrak{Z}_t(T(L_2(\mathbb{G}))^{**}, \Diamond) = L_1(\mathbb{G}) \oplus L_\infty(\mathbb{G})^\perp$$

In particular, for all infinite discrete groups, these equalities hold.

SAI and double commutation relations in $CB(L_\infty(\mathbb{G}))$

For the canonical embeddings

$$L_1(\mathbb{G}) \subseteq M(\mathbb{G}) \subseteq RM_{cb}(L_1(\mathbb{G})) \subseteq CB(L_\infty(\mathbb{G}))$$

Theorem

1. $L_1(\mathbb{G})^{cc} = M(\mathbb{G})^{cc} = RM_{cb}(L_1(\mathbb{G}))^{cc}$
2. $L_1(\mathbb{G})^{cc} = L_1(\mathbb{G}) \iff \mathbb{G}$ is discrete and $L_1(\mathbb{G})$ is SAI
3. $M(\mathbb{G})^{cc} = M(\mathbb{G}) \iff \mathbb{G}$ is co-amenable and $L_1(\mathbb{G})$ is SAI

Question

In $CB(L_\infty(\mathbb{G}))$, when will $RM_{cb}(L_1(\mathbb{G}))^{cc} = RM_{cb}(L_1(\mathbb{G}))$?

Or equivalently, when will $M(\mathbb{G})^{cc} = RM_{cb}(L_1(\mathbb{G}))$?

Double commutation relations in $CB(B(L_2(\mathbb{G})))$

For the canonical embeddings

$$L_1(\mathbb{G}) \subseteq M(\mathbb{G}) \subseteq RM_{cb}(L_1(\mathbb{G})) \cong RM_{cb}(T(L_2(\mathbb{G}))) \subseteq CB(B(L_2(\mathbb{G})))$$

we have $RM_{cb}(L_1(\mathbb{G}))^{cc} = RM_{cb}(L_1(\mathbb{G}))$ (Junge-Neufang-Ruan)

Corollary

1. $L_1(\mathbb{G})^{cc} = L_1(\mathbb{G}) \iff \mathbb{G}$ is discrete
2. $M(\mathbb{G})^{cc} = M(\mathbb{G}) \iff \mathbb{G}$ is co-amenable

Corollary

For $L_\infty(\mathbb{G}) = VN(SU(3))$, $L_1(\mathbb{G})^{cc} = L_1(\mathbb{G})$ in $CB(B(L_2(\mathbb{G})))$

but $L_1(\mathbb{G}) \subsetneq L_1(\mathbb{G})^{cc}$ in $CB(L_\infty(\mathbb{G}))$

Double commutation relations in $CB(B(L_2(\mathbb{G})))$

Note that $M(\mathbb{G}) \subseteq M(\mathbb{G})^{cc} \subseteq RM_{cb}(L_1(\mathbb{G}))$ in $CB(B(L_2(\mathbb{G})))$

Questions

1. Do we have

$$M(\mathbb{G})^{cc} = RM_{cb}(L_1(\mathbb{G})) \implies \mathbb{G} \text{ is co-amenable ?}$$

2. Do we have

$$M(G) \subseteq L_1(G)^{cc}$$

for non-discrete locally compact groups G?

Thank you.