Multi-norms, Weak compactness, and Amenability of Groups

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Background

Let (Ω, μ) be measure space. The following result was proved in a joint work with Garth Dales, Matt Daws, and Paul Ramsden[1]

Theorem

Suppose that $1 \leq p \leq q < \infty$. Then every (p, q)-multi-bounded subset of $L^1(\mu)$ is relatively weakly compact.

- The converse is not true.
- In fact, even the weak converse statement that "*every relatively weakly compact subset of $L^1(\mu)$ must be (p, q)-multi-bounded for some* $1 \leq p \leq q < \infty$" is not true.

[1] Multi-norms and the injectivity of $L^p(G)$, *J. London Math. Soc.* (2), 86 (2012), 779–809.

Background

Problem

Can weak compactness in $L^1(\mu)$ be characterized as a certain kind of boundedness using (p, q)-multi-norm?

- The answer is Yes and it works for $\mathfrak{L}^1$-spaces.
- Application to weakly compact operators from $\mathfrak{L}^\infty$-spaces.
- Application to amenability of locally compact groups.

Multi-norms

Let E be a Banach space. Multi-norms were introduced by Dales and Polyakov as follows.

A multi-norm is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ where

- each $\|\cdot\|_n$ is a norm on E^n and $\|\cdot\|_1 = \|\cdot\|$;
- $(\|\cdot\|_n : n \in \mathbb{N})$ satisfies the following axioms:
  1. $\|(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\|_n = \|(x_1, \ldots, x_n)\|_n \; \forall$ permutation σ;
  2. $\|(\alpha_1 x_1, \ldots, \alpha_n x_n)\|_n = \max_{1 \le i \le n} |\alpha_i| \, \|(x_1, \ldots, x_n)\|_n$;
  3. $\|(x_1, \ldots, x_{n-1}, 0)\|_n = \|(x_1, \ldots, x_{n-1})\|_{n-1}$;
  4. $\|(x_1, \ldots, x_{n-2}, x_{n-1}, x_{n-1})\|_n = \|(x_1, \ldots, x_{n-2}, x_{n-1})\|_{n-1}$.

The weak p-summing norms; $1 \leq p < +\infty$

Let E be a Banach space.

The weak p-summing norm on E^n is defined as

$$\mu_{p,n}(x_1, \ldots, x_n) := \sup \left\{ \left(\sum_{i=1}^{n} |\langle x_i, \lambda \rangle|^p \right)^{1/p} : \lambda \in E'_{[1]} \right\}$$

The weak p-summing norm on $(E')^n$ can also be computed as

$$\mu_{p,n}(\lambda_1, \ldots, \lambda_n) := \sup \left\{ \left(\sum_{i=1}^{n} |\langle x, \lambda_i \rangle|^p \right)^{1/p} : x \in E_{[1]} \right\}$$

The (p, q)-multi-norms; $1 \leq p, q < +\infty$

Define a norm $\|\cdot\|_n^{(p,q)}$ on E^n by

$$\|(x_1, \ldots, x_n)\|_n^{(p,q)} := \sup \left(\sum_{i=1}^{n} |\langle x_i, \lambda_i \rangle|^q \right)^{1/q}$$

taking over all $(\lambda_1, \ldots, \lambda_n) \in (E')^n$ with $\mu_{p,n}(\lambda_1, \ldots, \lambda_n) \leq 1$.

When $p \leq q$, $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ is a multi-norm on E, and is called the (p, q)-multi-norm.

Multi-bounded sets

Let E be a Banach space, and let B be a subset of E.

The set B is multi-bounded with respect to a multi-norm $(\|\cdot\|_n\colon\ n\in\mathbb{N})$ iff

$$\sup\{\|(x_1,\dots,x_n)\|_n\colon\ x_1,\dots,x_n\in B,\ n\in\mathbb{N}\}<+\infty.$$

More specifically:

Definition

When $p\le q\in[1,\infty)$, the set B is (p,q)-multi-bounded if

$$\sup\left\{\|(x_1,\dots,x_n)\|_n^{(p,q)}\colon\ x_1,\dots,x_n\in B,\ n\in\mathbb{N}\right\}<+\infty.$$

The concept of (p,q)-multi-bounded makes little sense when $q<p$.

Absolute (q, p)-summing operators

Let E and F be Banach spaces.
Let $T : E \to F$ be a linear operator.
Let $p, q \in [1, \infty)$. The n^{th} (q, p)-summing constant of T is

$$\pi_{q,p}^{(n)}(T) := \sup \left(\sum_{i=1}^{n} \| Tx_i \|^{q} \right)^{1/q}$$

taking over all $x_1, \ldots, x_n \in E$ with $\mu_{p,n}(x_1, \ldots, x_n) \leq 1$.
The operator T is absolutely (q, p)-summing if

$$\pi_{q,p}(T) := \sup_{n \in \mathbb{N}} \pi_{q,p}^{(n)}(T) < \infty \, .$$

When $q < p$, the only absolutely (q, p)-summing operator is the zero operator.

A relation with (q, p)-multi-bounded

Let E be a Banach space, and let $(x_n)_{n=1}^{\infty} \subseteq E$ be bounded. Then $T(\delta_n) := x_n$ extends to a bounded linear operator $T : \ell^1 \to E$, and we have

(x_n) is (p, q)-multi-bounded if and only if $T' : E' \to \ell^{\infty}$ is absolutely (q, p)-summing.

Theorem (Kwapień–Pełczyński)

$\Sigma : \ell^1 \to \ell^{\infty}, (\alpha_n) \mapsto (\sum_{i=1}^{n} \alpha_i)_n$ is (q, p)-summing for every $1 \leq p < q < +\infty$, but is not weakly compact.

Consequently:

Theorem (Dales, Daws, P, Ramsden)

Take $p \in [1, \infty)$. Then every (p, p)-multi-bounded subset of E is relatively weakly compact.
But there is a subset in c_0 that is (p, q)-multi-bounded for every $1 \leq p < q < \infty$, but is not relatively weakly compact.

The (p, q)-multi-norms on $L^1(\mu)$; $1 \leq p \leq q < +\infty$

Let (Ω, μ) be a measure space.

Theorem (Pisier)

Suppose that $p < q < r \in [1, \infty)$. Then the class of absolutely (q, p)-summing operators from $L^\infty(\mu)$ is the same as the class of absolutely $(q, 1)$-summing operators from $L^\infty(\mu)$, and is contained in the class of absolutely (r, r)-summing operators.

Theorem (Dales, Daws, P, Ramsden)

Suppose that $p < q < r \in [1, \infty)$. Then, on $L^1(\mu)$, the (p, q)-multi-norm and the $(1, q)$-multi-norm are equivalent, and they majorize the (r, r)-multi-norm.

Corollary

Suppose that $p \leq q \in [1, \infty)$. Then, on $L^1(\mu)$, every (p, q)-multi-bounded set is relatively weakly compact.

Relatively compact, non (p, q)-multi-bounded in $L^1(\mu)$

For each $k \in \mathbb{N}$, define

$$g_k(t) := \begin{cases} 1/\ln t & \text{if } t \in [k+1, k+2] \\ 0 & \text{otherwise.} \end{cases}$$

Consider $B = \{g_k : \; k \in \mathbb{N}\}$.
Then

$$\|g_k\|_{L^1} = \int_{k+1}^{k+2} \frac{\mathrm{d}t}{\ln t} \leq \frac{1}{\ln(k+1)} \to 0 \quad \text{as} \quad k \to \infty \, .$$

Hence B is actually a relatively compact subset of $L^1(\mathbb{R})$.

On the other hand,

$$\|(g_1, \ldots, g_n)\|_n^{(1,1)} = \int_2^{n+2} \frac{\mathrm{d}t}{\ln t}\,.$$

For each $p, q \in [1, \infty)$,

$$\begin{aligned}\|(g_1, \ldots, g_n)\|_n^{(p,q)} &\geq \|(g_1, \ldots, g_n)\|_n^{(1,q)} \\ &\geq n^{1/q-1}\|(g_1, \ldots, g_n)\|_n^{(1,1)} \\ &= \frac{\int_2^{n+2} \mathrm{d}t/\ln t}{n^{1-1/q}} \to \infty \quad \text{as} \quad n \to \infty\,.\end{aligned}$$

Thus B is not (p, q)-multi-bounded for any $p, q \in [1, \infty)$.

The Dunford and Pettis theorem

Let $B \subset L^1(\mu)$ be bounded. Then the following are equivalent[2]:

1. B is relatively weakly compact.
2. B is uniformly integrable.
3. For every disjoint measurable subsets $X_1, X_2, \dots$ of Ω

$$\inf_{n\in\mathbb{N}} \sup_{f\in B} \int_{X_n} |f| \,\mathrm{d}\mu = 0\,;$$

4. There is *no* basic sequence in B that is equivalent to the standard basis of ℓ^1.
5. For every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that, for every pairwise disjoint measurable subsets $X_1, \dots, X_n$ of Ω,

$$\min_{1\le k\le n} \sup \left\{ \int_{X_k} |f| \,\mathrm{d}\mu \colon f \in B \right\} \le \varepsilon\,;$$

6. $\forall K > 0$, $\exists$ an $n \in \mathbb{N}$ such that there is *no* $f_1, \dots, f_n$ in B that are K-equivalent to the standard basis of ℓ^1_n.

[2] Wojtaszczyk, *Banach spaces for analysts*

The (p, q)-multi-norms on $L^1(\mu)$; $1 \le p < q < +\infty$

Theorem (Dales, Polyakov)

Suppose that $q \in [1, \infty)$. Then, on $L^1(\mu)$, the $(1, q)$-multi-norm is equal the standard q-multi-norm, which is defined as follows.

For each $f_1, \dots, f_n \in L^1(\mu)$

$$\|(f_1, \dots, f_n)\|_n^{[q]} := \sup \left(\sum_{i=1}^{n} \|f_i|X_i\|_{L^1}^q \right)^{1/q},$$

where the supremum is taken over all measurable partitions $\{X_1, \dots, X_n\}$ of Ω.

Almost (p, q)-multi-boundedness; $1 \leq p, q < \infty$

A subset B of a Banach space E is almost (p, q)-multi-bounded if it satisfies either one of the following four *equivalent* conditions:

(a) $\inf_{n \in \mathbb{N}} \dfrac{\|(x_1, \ldots, x_n)\|_n^{(p,q)}}{n^{1/q}} = 0$ for every sequence (x_n) in B;

(b) $\lim_{n \to \infty} \dfrac{\|(x_1, \ldots, x_n)\|_n^{(p,q)}}{n^{1/q}} = 0$ for every sequence (x_n) in B;

(c) for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\frac{\|(x_1, \ldots, x_n)\|_n^{(p,q)}}{n^{1/q}} \leq \varepsilon \quad (x_1, \ldots, x_n \in B)\,;$$

(d) for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that

$$\frac{\|(x_1, \ldots, x_m)\|_m^{(p,q)}}{m^{1/q}} \leq \varepsilon \quad (x_1, \ldots, x_m \in B\,,\ m \geq n)$$

Let E be a Banach space, and let $B \subseteq E$. Then we always have

$$
\begin{aligned}
d_{p,q}(B) := \sup \left\{ \inf_{n\in\mathbb{N}} \frac{\|(x_1,\ldots,x_n)\|_n^{(p,q)}}{n^{1/q}} : \ (x_n) \subset B \right\} \\
\le c_{p,q}(B) := \sup \left\{ \limsup_{n\to\infty} \frac{\|(x_1,\ldots,x_n)\|_n^{(p,q)}}{n^{1/q}} : \ (x_n) \subset B \right\} \\
= \inf_{n\in\mathbb{N}} \ \sup \left\{ \frac{\|(x_1,\ldots,x_n)\|_n^{(p,q)}}{n^{1/q}} : \ x_1,\ldots,x_n \in B \right\} \\
= \lim_{n\to\infty} \ \sup \left\{ \frac{\|(x_1,\ldots,x_n)\|_n^{(p,q)}}{n^{1/q}} : \ x_1,\ldots,x_n \in B \right\} \\
\le 2^{1/p} d_{p,q}(B)\,.
\end{aligned}
$$

Moreover, the functions $(p,q) \mapsto c_{p,q}(B)$ and $(p,q) \mapsto d_{p,q}(B)$ are increasing.

Almost (q, p)-multi-bounded sets in $L^1(\mu)$

Let (Ω, μ) be a measure space. Suppose that $p, q \in [1, \infty)$.

Theorem

A subset of $L^1(\mu)$ is almost (p, q)-multi-bounded if and only if it is relatively weakly compact.

Corollary

A subset B of $L^1(\mu)$ is relatively weakly compact if and only if

$$\lim_{n\to\infty} \frac{\| \, |f_1| \vee \cdots \vee |f_n| \, \|_{L^1}}{n} = 0 \quad \textit{for every } (f_n) \subseteq B.$$

In general Banach spaces

- There is a subset of c_0 that is almost (p, q)-multi-bounded for every $p, q \in [1, \infty)$, but is not relatively weakly compact.
- The closed unit ball B of ℓ^2-$\bigoplus_{n=1}^{\infty} \ell_n^1$ is weakly compact, but has $c_{p,q}(B) \geq 1$ (and thus, $c_{p,q}(B) = 1$), so that B is not almost (p, q)-multi-bounded whenever $p, q \in [1, \infty)$.

Lindenstrauss and Pełczyński's $\mathfrak{L}^r$-spaces

A Banach space E is called an $\mathfrak{L}^r$-space if there is a $\lambda > 1$ such that, for any $M \subseteq_{FIN} E$, there is $N \subseteq_{FIN} E$ containing M and an isomorphism $T : N \to \ell^r_{\dim N}$ such that $\|T\|\|T^{-1}\| \leq \lambda$.

Theorem (Lindenstrauss and Rosenthal)

Suppose that $r = 1$ or $r = \infty$. Then a Banach space E is an $\mathfrak{L}^r$-space if and only if there is a $\lambda > 0$ such that, for any $M \subseteq_{FIN} E$, the inclusion $M \hookrightarrow E$ can be factored as TS through ℓ^r_n for some $n \in \mathbb{N}$ with $\|T\|\|S\| \leq \lambda$.

Examples of $\mathfrak{L}^r$-spaces include the spaces $L^r(\mu)$ as well as their complemented subpaces. The class of $\mathfrak{L}^\infty$-spaces also includes all $\mathcal{C}(K)$-spaces and their closed sublattices.

Theorem (Lindenstrauss and Rosenthal)

A Banach space E is an $\mathfrak{L}^r$-space if and only if E'', which is just E if $1 < r < \infty$, is a complemented subspace of $L^r(\Omega)$ for some measure space Ω.

Theorem (Lindenstrauss and Rosenthal)

A Banach space E is an $\mathfrak{L}^r$-space if and only if E' is an $\mathfrak{L}^{r'}$-space, where $r' \in [1, \infty]$ is conjugate to r.

Almost (q, p)-multi-bounded sets in $\mathfrak{L}^1$-spaces and operators from $\mathfrak{L}^\infty$-spaces

Theorem

Let E be an $\mathfrak{L}^1$-space, and let $p, q \in [1, \infty)$. Then a subset of E is relatively weakly compact if and only if it is almost (p, q)-multi-bounded.

Theorem

Let E be an $\mathfrak{L}^\infty$-space, let F be a Banach space, and let $p, q \in [1, \infty)$. Then a linear operator $T : E \to F$ is weakly compact if and only if

$$\lim_{n\to\infty} \frac{\pi_{q,p}^{(n)}(T)}{n^{1/q}} = 0\,.$$

Let $n \in \mathbb{N}$, and denote $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_n)$ and $\boldsymbol{x} := (x_1, \ldots, x_n)$. Then

$$\begin{aligned}
&\sup\left\{\|(f_1, \ldots, f_n)\|_n^{(p,q)} :\ f_1, \ldots, f_n \in T'[F'_{[1]}]\right\} \\
= \quad & \sup\left\{\|(T'\lambda_1, \ldots, T'\lambda_n)\|_n^{(p,q)} :\ \boldsymbol{\lambda} \in (F'_{[1]})^n\right\} \\
= \quad & \sup\left\{\left(\sum_{i=1}^n \left|\left\langle x_i, T'\lambda_i\right\rangle\right|^q\right)^{1/q} :\ \boldsymbol{\lambda} \in (F'_{[1]})^n,\ \boldsymbol{x} \in E^n,\ \mu_{p,n}(\boldsymbol{x}) \leq 1\right\} \\
= \quad & \sup\left\{\left(\sum_{i=1}^n |\langle Tx_i, \lambda_i\rangle|^q\right)^{1/q} :\ \boldsymbol{\lambda} \in (F'_{[1]})^n,\ \boldsymbol{x} \in E^n,\ \mu_{p,n}(\boldsymbol{x}) \leq 1\right\} \\
= \quad & \sup\left\{\left(\sum_{i=1}^n \|Tx_i\|^q\right)^{1/q} :\ \boldsymbol{x} \in E^n,\ \mu_{p,n}(\boldsymbol{x}) \leq 1\right\} = \pi_{q,p}^{(n)}(T)
\end{aligned}$$

Amenability of locally compact groups

Let G be a locally compact group.

- G is amenable if $\exists$ a left invariant mean Λ on $L^\infty(G)$ i.e. iff $\exists$ a mean Λ such that $\{s \cdot \Lambda \colon s \in G\} = \{\Lambda\}$.
- By the Ryll-Nardzewski fixed point theorem, G is amenable if and only if $\{s \cdot \Lambda \colon s \in G\}$ is relatively weakly compact in $L^\infty(G)'$.
- $L^\infty(G)'$ is an $\mathfrak{L}^1$-space.
- G is amenable if and only if $\{s \cdot \Lambda \colon s \in G\}$ is almost (p, q)-multi-bounded in $L^\infty(G)'$.

The standard Følner's conditions

(FC): For every $\varepsilon > 0$ and every finite subset $F \subset G$, there exists a compact subset $S \subseteq G$ such that

$$m(tS\Delta S) < \varepsilon m(S) \qquad (\forall t \in F).$$

Where $A\Delta B := (A \setminus B) \cup (B \setminus A)$.

(WFC): The above holds for a single $\varepsilon \in (0, 2)$.

(SFC): For every compact subset $K \subset G$ and every $\varepsilon > 0$, there exists a compact subset $S \subseteq G$ such that

$$m(KS\Delta S) < \varepsilon m(S)\,.$$

(WFC) $\Leftrightarrow$ (FC) $\Leftrightarrow$ (SFC) $\Leftrightarrow$ Amenability of G

A (new) Følner-type condition

(NFC) There exist $\varepsilon \in (0,1)$ and $n_0 \in \mathbb{N}$ such that, for every finite subset $F \subset G$ with $|F| \geq n_0$, there exists a compact subset $S \subseteq G$ such that

$$m(ES) < \varepsilon\,|E|\,m(S) \qquad (\forall E \subseteq F \text{ with } |E| \geq n_0).$$

Theorem

(NFC) $\Leftrightarrow$ Amenability of G $\quad$ ($\Leftrightarrow$(WFC) $\Leftrightarrow$ (FC) $\Leftrightarrow$ (SFC))

(WFC) $\Rightarrow$ (NFC)

Let $0 < \varepsilon_0 < 2$ be the number in (WFC).

Let F be any finite subsets of G.

By (WFC), there exists a compact subset $S \subseteq G$ such that

$$\frac{m(tS\Delta S)}{m(S)} < \varepsilon_0 \qquad (\forall t \in F).$$

It follows that if $E \subseteq F$ with $|E| \geq n_0$, then

$$\frac{m(ES)}{m(S)} < 1 + \frac{\varepsilon_0}{2}|E| \leq \delta_0 |E|$$

when we choose δ_0 with $\varepsilon_0/2 < \delta_0 < 1$ and choose $n_0 \in \mathbb{N}$, $n_0 > 2/(2\delta_0 - \varepsilon_0)$.

A little stronger version of (NFC)

(SNFC) For every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that, for every finite subset $F \subset G$ with $|F| \geq n_\varepsilon$, there exists a compact subset $S \subseteq G$ such that

$$m(ES) < \varepsilon\,|E|\,m(S) \qquad (\forall E \subseteq F \text{ with } |E| \geq n_\varepsilon).$$

Obviously $\qquad$ (FC) $\Rightarrow$ (SNFC) $\Rightarrow$ (NFC)

Proof of "(SNFC) $\Rightarrow$ G is amenable"

For $\varepsilon > 0$ and finite subset $F \subseteq G$ with $|F| \geq n_\varepsilon$, set $a_{\varepsilon,F} := \frac{1}{m(S)}\chi_S$.

Then, for every $\{t_1, \ldots, t_n\} \subseteq F$ with $|\{t_1, \ldots, t_n\}| \geq n_\varepsilon$

$$\|(t_1 \cdot a_{\varepsilon,F}, \ldots, t_n \cdot a_{\varepsilon,F})\|_n^{(1,1)} = \frac{m(\{t_1, \ldots, t_n\}\, S)}{m(S)} \leq \varepsilon n$$

Let $\Lambda \in L^\infty(G)'$ be a weak* limit of $(a_{\varepsilon,F})$.

We see that, for every $(t_n) \subset G$,

$$\lim_{n\to\infty} \frac{\|(t_1 \cdot \Lambda, \ldots, t_n \cdot \Lambda)\|_n^{(1,1)}}{n} = 0.$$

$\ldots$

Hence, G is amenable.

Pseudo-amenability

Definition (Dales–Polyakov)

G is pseudo-amenable iff
(PA) For every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for every finite subset $F \subset G$ with $|F| \geq n_\varepsilon$, there exists a compact subset $S \subseteq G$ such that

$$m(FS) < \varepsilon\,|F|\,m(S).$$

It is obvious that (SNFC) $\Rightarrow$ Pseudo-amenability.

Question

Pseudo-amenability of $G \Longrightarrow$ Amenability of G?

Weak pseudo-amenability

(WPA) There exists $0 < \varepsilon_0 < 1$ and $n_0 \in \mathbb{N}$ such that, for every finite subset $F \subset G$ with $|F| \geq n_0$, there exists a compact subset $S \subseteq G$ such that

$$m(FS) < \varepsilon_0 \, |F| \, m(S).$$

Question

(WPA) $\Longrightarrow$ Amenability of G?