Small Deformations of Algebras of Analytic Functions in C^n

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If "similar" $=$ both $\mathbb{C}\backslash\Omega_1$ and $\mathbb{C}\backslash\Omega_2$ have two connected components then the domains may not be holomorphically equivalent but they are equivalent to $\{z : r_1 < |z| < 1\}$ and $\{z : r_2 < |z| < 1\}$ for some r_1, r_2; hence they are holomorphically equivalent to domains that "look almost equivalent".

Definitions

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A Banach algebra $\mathcal{B}$ is a **metric δ-deformation** of $\mathcal{A}$ if there is a $T : \mathcal{A} \to \mathcal{B}$ such that $\|T\| \left\|T^{-1}\right\| \leq 1 + \delta$ (and $Te = e$).

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Definition

A new multiplication $\times$ defined on the same Banach space $\mathcal{A}$ is an **algebraic δ-deformation** of $(\mathcal{A}, \cdot)$ if $\|\times - \cdot\| \leq \delta$; that is, if

$$\|a \cdot b - a \times b\| \leq \delta \|a\| \|b\|, \qquad \text{for } a, b \in \mathcal{A}$$

(and $e_\times = e$).

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Example

Put $P_\varepsilon = \{z : 1 \le |z| \le 2 + \varepsilon\}$, and define $T_\varepsilon : A(P_0) \to A(P_\varepsilon)$ by

$$T_\varepsilon\left(\sum_{n=-\infty}^{\infty} a_n z^n\right) = \sum_{n=-\infty}^{0} a_n z^n + \sum_{n=1}^{\infty} a_n \left(\frac{2}{2+\varepsilon}\right)^n z^n.$$

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- Deformations of topological algebras, latices, other function spaces; non-linear deformations, etc.

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Theorem (Nagasawa, 1959)

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Theorem (Cambern, 1963)

$C(X) \approx C(Y) \Longrightarrow X \approx Y$

$d_{B-M}(C(X), C(Y)) < 2 \Longrightarrow X, Y$ *are homeomorphic.*

Definition

$d_{B-M}(\mathcal{A}, \mathcal{B}) = \inf\left\{\|T\| \left\|T^{-1}\right\| : T : \mathcal{A} \to \mathcal{B}\right\}.$

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Theorem (B.E. Johnson and I. Raeburn & J. Taylor, 1977)

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Theorem (RR, 1973, 75, 77)

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$\mathcal{A} = A(S)$, S *a 1-dim Riemann surface then*
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Corollary

The disc algebra $A\left(\mathbb{D}\right)$ *is the only stable* $A(S)$ *algebra.*

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Problem

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Very little is known about the multidimensional case - only partial results concerning $\mathcal{A}(\mathbb{D}^n)$ (KJ, 1992) and $\mathcal{A}(\mathbb{B}^n)$ (KJ, 2012).

Deformation of the ball algebra

Definition

The Ball Algebra $\mathcal{A}(B_n)$ is the algebra of all continuos functions on the closed unit ball $\overline{B}_n$ which are analytic in B_n.

Theorem

Assume that $\mathcal{B}$ is a Banach algebra such that $d_{B-M}(\mathcal{A}(B_n), B) < 1 + \varepsilon_0$ then there is a complex manifold Ω, homeomorphic with B_n and such that $\mathcal{B} = \mathcal{A}(\Omega)$.

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Problem

It is not known if Ω *must be holomorphically equivalent to* B_n.

Proof - the basic idea

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- show that we can introduce an analytic structure on $\mathfrak{M}(\mathcal{B})$ such that all functions from $\mathcal{B}$ are analytic.

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Let $\mathcal{A}$ be a complex uniform algebra then the following conditions are equivalent

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(we assume that all multiplications have the same unit).

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   it follows that $\mathcal{B}$ is a uniform algebra and that the Chauquet boundaries of $\mathcal{A}$ and $\mathcal{B}$ are homeomorphic.

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Definition

$\mathcal{A}$ is *functionally-stable* or *f-stable* (or *AMNM* algebra) if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall F \in \mathfrak{M}_\delta(\mathcal{A}) \; \exists G \in \mathfrak{M}(\mathcal{A}) \;\; \|F - G\| \leq \varepsilon,$$

where $\mathfrak{M}_\delta(\mathcal{A})$ is the set of δ-multiplicative functionals on $\mathcal{A}$.

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If $f_0 \in L^1(0,1)$ is "near" the Dirac measure then $f \longmapsto f * f_0$ is almost multiplicative.

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Example (S.J. Sidney, 1997)

There exists a non f-stable uniform algebra.

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Theorem

- *The ball algebras $\mathcal{A}(B_n)$ are f-stable.*

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Let $B(z) = \prod_{n=1}^{\infty} \frac{\bar{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z}$ be a Blaschke product. Then

- *if B is an interpolating Blaschke product then H^∞ / BH^∞ is f-stable;*

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- *if B is an interpolating Blaschke product then H^{∞} / BH^{∞} is f-stable;*
- *if B is not a product of finitely many interpolating Blaschke product then H^{∞} / BH^{∞} is not f-stable.*

Almost multiplicative functionals - recent results

Theorem

- *The ball algebras $\mathcal{A}(B_n)$ are f-stable.*
- *Any uniform algebra with one generator is f-stable.*
- *If $K \subset \mathbb{C}$ is such that $\mathbb{C} \setminus K$ has finitely many components and the closures of the components are disjoint then $R(K)$ is f-stable.*

Theorem

Let $B(z) = \prod_{n=1}^{\infty} \frac{\bar{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z}$ be a Blaschke product. Then

- *if B is an interpolating Blaschke product then H^∞ / BH^∞ is f-stable;*
- *if B is not a product of finitely many interpolating Blaschke product then H^∞ / BH^∞ is not f-stable.*

Problem

Is $H^\infty(\mathbb{D})$ is f-stable? (Does $H^\infty(\mathbb{D})$ have an almost corona?)

Almost multiplicative functionals - proof

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Proof. Let $F \in \mathfrak{M}_\delta\left(\mathcal{A}(B_n)\right)$. We show that F is near a multiplicative functional.

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Step 1. Show that (without loss of generality) F may be represented by a probabilistic, nonatomic measure μ_F on ∂B_n.

Step 2. For $\mathbf{w} = (w_1, ..., w_n) \in B_n$ define $\Phi_\mathbf{w} : \bar{B}_n \to \bar{B}_n$ by

$$\Phi_\mathbf{w}(\mathbf{z}) = \frac{\mathbf{w} - P_\mathbf{z} - \sqrt{1 - \|\mathbf{w}\|^2}\,(\mathbf{z} - P_\mathbf{z})}{1 - \langle \mathbf{z}, \mathbf{w} \rangle},$$

where

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{k=1}^{n} z_k \bar{w}_k \quad \text{and} \quad P_\mathbf{z} = \frac{\langle \mathbf{z}, \mathbf{w} \rangle}{\|\mathbf{w}\|^2}\mathbf{w}.$$

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Step 3. Define $\varphi : B_n \longrightarrow B_n$ by

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$$B_n \ni \mathbf{w}_j \longrightarrow \mathbf{w}_0 \in \partial B_n,$$

then

$$\Phi_{\mathbf{w}_j}(z) \longrightarrow \mathbf{w}_0 \quad \text{pointwise on } \bar{B}_n \backslash \{\mathbf{w}_0\}.$$

Since μ_F has no atoms, φ extends to a continuous function on $\bar{B}_n$ such that $\varphi(\mathbf{w}) = \mathbf{w}$ for $\mathbf{w} \in \partial B_n$.

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$$F(z_k) = 0 \quad \text{for } k = 1, ..., n.$$

Step 4. Put $\mathcal{A}_0(B_n) \overset{df}{=} \{f \in \mathcal{A}(B_n) : f(\mathbf{0}) = 0\}$ and define

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As a special case, for $n = 1$ we get Johnson's theorem.

Open problems - deformations of uniform algebras

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For bounded pseudoconvex domains Ω in C^n describe small deformations of algebras $A(\Omega)$ and $H^\infty(\Omega)$ and characterize these domains for which the algebras are stable.

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Is the property "$\mathcal{A}$ has n generators" for $n \in \mathbb{N} \cup \{\infty\}$ stable?

Open problems - almost multiplicative functionals

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Describe small deformations of topological algebras of analytic functions on bounded pseudoconvex domains Ω *in* C^n

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Fact

For $n = 1$ there are partial results available, similar to Rochberg's 1986 description of deformation of $\mathcal{A}(S)$ [M. Abel & KJ, 2003]

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Problem

Describe small deformations of real uniform algebras.

More open problems - separating maps

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S may not be continuous (even if Y is a singleton), if S^{-1} exists then S is continuous and S^{-1} is separating.

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Theorem (J. Araujo & KJ, 1999)

Yes if $X\subset\mathbb{R}$.

Thanks!

- THANK YOU!