Small Deformations of Algebras of Analytic Functions in $C^n$

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Problem

Assume domains $\Omega_1$ and $\Omega_2$ are "similar". Does it follow that they are "almost holomorphically equivalent", or even better holomorphically equivalent?
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Example

If "similar" = both $\Omega_1$ and $\Omega_2$ are simply connected in $\mathbb{C}^1$ then indeed it follows that they are holomorphically equivalent.
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Example

If "similar" = both $\mathbb{C} \setminus \Omega_1$ and $\mathbb{C} \setminus \Omega_2$ have two connected components then the domains may not be holomorphically equivalent but they are equivalent to $\{z : r_1 < |z| < 1\}$ and $\{z : r_2 < |z| < 1\}$ for some $r_1, r_2$; hence they are holomorphically equivalent to domains that "look almost equivalent".
A Banach algebra $B$ is a \textbf{metric $\delta$-deformation} of $A$ if there is a $T : A \to B$ such that $\| T \| \| T^{-1} \| \leq 1 + \delta$ (and $Te = e$).
Definition

A Banach algebra $\mathcal{B}$ is a **metric $\delta$-deformation** of $\mathcal{A}$ if there is a $T : \mathcal{A} \to \mathcal{B}$ such that $\| T \| \| T^{-1} \| \leq 1 + \delta$ (and $Te = e$).

Definition

A new multiplication $\times$ defined on the same Banach space $\mathcal{A}$ is an **algebraic $\delta$-deformation** of $(\mathcal{A}, \cdot)$ if $\| \times - \cdot \| \leq \delta$; that is, if

$$\| a \cdot b - a \times b \| \leq \delta \| a \| \| b \| , \quad \text{for } a, b \in \mathcal{A}$$

(and $e_\times = e$).
Example (trivial)

Put $T = S + \Delta$ where $S : \mathcal{A} \to \mathcal{B}$ is an isometric isomorphism and $\Delta : \mathcal{A} \to \mathcal{B}$ has small norm.
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a \times b \overset{df}{=} T^{-1} (Ta \cdot Tb).
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_Under what circumstances any small deformation can be defined by the above formula with \( T = \text{Id} + \Delta : \mathcal{A} \to \mathcal{A} \)?_
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*Under what circumstances any small deformation can be defined by the above formula with $T = \text{Id} + \Delta : \mathcal{A} \to \mathcal{A}$? (\mathcal{A} \text{ is strongly stable})*
**Examples**

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**Problem**

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**Example**

Put $P_\varepsilon = \{ z : 1 \leq |z| \leq 2 + \varepsilon \}$, and define $T_\varepsilon : A(P_0) \to A(P_\varepsilon)$ by

\[
 T_\varepsilon \left( \sum_{n=-\infty}^{\infty} a_n z^n \right) = \sum_{n=-\infty}^{0} a_n z^n + \sum_{n=1}^{\infty} a_n \left( \frac{2}{2 + \varepsilon} \right)^n z^n .
\]
Problem

- $A \approx B \Rightarrow A = B$ (A is stable).
\begin{itemize}
  \item $A \cong B \not\Rightarrow A = B$ ($A$ is stable).
  
  \item Describe all small deformations of $A$. Is there a continuous (analytic) structure on that family of small deformations?
\end{itemize}
Problems

- $\mathcal{A} \cong \mathcal{B} \Rightarrow \mathcal{A} = \mathcal{B}$ ($\mathcal{A}$ is stable).
- Describe all small deformations of $\mathcal{A}$. Is there a continuous (analytic) structure on that family of small deformations?
- $\mathcal{A} \cong \mathcal{B} \Rightarrow \mathcal{A}$ and $\mathcal{B}$ share the same properties (e.g.: Dirichlet, logmodular, finitely generated, has analytic structure in the spectrum, etc.).
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- Deformation of complex manifolds and multidimensional Riemann Mapping Theorem.
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- Almost multiplicative functionals.
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- Deformation of complex manifolds and multidimensional Riemann Mapping Theorem.
- Almost multiplicative functionals.
- Deformations of topological algebras, lattices, other function spaces; non-linear deformations, etc.
Theorem (Nagasawa, 1959)

Uniform algebras are isometric iff they are isomorphic as algebras.
Very Early History (metric case)

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*Uniform algebras are isometric iff they are isomorphic as algebras.*

**Definition**

\[ \mathcal{A} \text{ is a uniform algebra iff } \|a^2\| = \|a\|^2 \text{ or equivalently } \mathcal{A} \subset C(X). \]
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Not true for Banach algebras in general: \( (l^1, \cdot), (l^1, *) \).
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Theorem (Cambern, 1963)

\( C(X) \cong C(Y) \implies X \cong Y \)
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**Example**

Not true for Banach algebras in general: $(l^1, \cdot)$, $(l^1, *)$.

**Theorem (Cambern, 1963)**

$C(X) \approx C(Y) \implies X \approx Y$

$d_{B-M}(C(X), C(Y)) < 2 \implies X, Y$ are homeomorphic.

**Definition**

$d_{B-M}(\mathcal{A}, \mathcal{B}) = \inf \left\{ \|T\| \|T^{-1}\| : T : \mathcal{A} \to \mathcal{B} \right\}$. 
Early History

**Theorem (B.E. Johnson and I. Raeburn & J. Taylor, 1977)**

A Banach algebra $\mathcal{B}$ is algebraically strongly stable if Hochschild cohomology groups $H^2(\mathcal{B}, \mathcal{B})$ and $H^3(\mathcal{B}, \mathcal{B})$ vanish.
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A Banach algebra $\mathcal{B}$ is algebraically strongly stable if Hochschild cohomology groups $\mathcal{H}^2(\mathcal{B}, \mathcal{B})$ and $\mathcal{H}^3(\mathcal{B}, \mathcal{B})$ vanish.
Theorem (RR, 1973, 75, 77)

\[ d_{B-M}(A \subset C(X), B \subset C(Y)) < 1 + \epsilon \iff X, Y \text{ are homeomorphic.} \]
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*Metric and algebraic deformations coincide for many uniform algebras.*
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The disc algebra \( A(D) \) is stable.
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**Theorem (RR, 1986)**

\[
\mathcal{A} = A(S), \ S \text{ a 1-dim Riemann surface then } \quad d_{B-M}(\mathcal{A}, \mathcal{B}) \approx d_{\text{quasiconformal}}(S, S'), \text{ automatically } \mathcal{B} = A(S').
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**Theorem (RR, 1973, 75, 77)**

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**Theorem (RR, 1986)**

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\[ d_{B-M}(\mathcal{A}, \mathcal{B}) \approx d_{\text{quasiconformal}}(S, S'), \text{ automatically } \mathcal{B} = A(S'). \]

**Corollary**

*The disc algebra \( A(D) \) is the only stable \( A(S) \) algebra.*
Problem

Describe small deformations of Banach algebras of analytic functions defined on $n$-dim manifolds, $n > 1$. 

Deformations of complex manifolds
Deformations of complex manifolds

**Problem**

Describe small deformations of Banach algebras of analytic functions defined on $n$-dim manifolds, $n > 1$.

**Definition**

$$d(\Omega, \Omega') = \inf \{ \|T\| \|T^{-1}\| : T : A(\Omega) \to A(\Omega') \}.$$
Deformations of complex manifolds

Problem

Describe small deformations of Banach algebras of analytic functions defined on n-dim manifolds, \( n > 1 \).

Definition

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d (\Omega, \Omega') = \inf \left\{ \| T \| \| T^{-1} \| : \quad T : A(\Omega) \rightarrow A(\Omega') \right\}.
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Problem

Can the Rochberg's result be extended to domains in \( \mathbb{C}^n \), \( n > 1 \) to provide a quantitative multidimensional Riemann Mapping Theorem?
Deformations of complex manifolds

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Describe small deformations of Banach algebras of analytic functions defined on n-dim manifolds, $n > 1$.

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Problem

Can the Rochberg’s result be extended to domains in $\mathbb{C}^n$, $n > 1$ to provide a quantitative multidimensional Riemann Mapping Theorem? Very little is known about the multidimensional case - only partial results concerning $A(D^n)$ (KJ, 1992) and $A(B^n)$ (KJ, 2012).
The Ball Algebra $\mathcal{A}(B_n)$ is the algebra of all continuous functions on the closed unit ball $B_n$ which are analytic in $B_n$.

**Theorem**

Assume that $\mathcal{B}$ is a Banach algebra such that $d_{BM}(\mathcal{A}(B_n), \mathcal{B}) < 1 + \varepsilon_0$ then there is a complex manifold $\Omega$, homeomorphic with $B_n$ and such that $\mathcal{B} = \mathcal{A}(\Omega)$. 
Deformation of the ball algebra

Definition

The Ball Algebra $\mathcal{A}(B_n)$ is the algebra of all continuous functions on the closed unit ball $\overline{B}_n$ which are analytic in $B_n$.

Theorem

Assume that $\mathcal{B}$ is a Banach algebra such that $d_{B-M}(\mathcal{A}(B_n), \mathcal{B}) < 1 + \varepsilon_0$ then there is a complex manifold $\Omega$, homeomorphic with $B_n$ and such that $\mathcal{B} = \mathcal{A}(\Omega)$.

Problem

It is not known if $\Omega$ must be holomorphically equivalent to $B_n$. 

Krzysztof Jarosz (Southern Illinois University) Deformations of Banach Algebras 10/24
Proof - the basic idea

Take an almost isometry $T : A(B_n) \to B; \|T\| \|T^{-1}\| < 1 + \varepsilon_0$
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Take an almost isometry \( T : \mathcal{A}(B_n) \to \mathcal{B} ; \| T \| \| T^{-1} \| < 1 + \varepsilon_0 \)

- show that \( \mathcal{B} \) must automatically be a uniform algebra
Proof - the basic idea

Take an almost isometry \( T : \mathcal{A}(B_n) \to \mathcal{B} ; \| T \| \| T^{-1} \| < 1 + \varepsilon_0 \)

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- show that without loss of generality we may assume \( T1 = 1 \)
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- show that $\mathcal{B}$ must automatically be a uniform algebra
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- take a linear and multiplicative functional $F$ on $\mathcal{B}$ and define a functional on $\mathcal{A}(B_n)$ by $G = F \circ T$
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- show that $G$ is almost multiplicative, that is
  \[ |F(fg) - F(f)F(g)| \leq \delta \|f\| \|g\| \]
  for some small $\delta$ dependent only on $\varepsilon_0$
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- show that any almost multiplicative functional $G$ on $\mathcal{A}(B_n)$ must be close to a multiplicative functional $\tilde{G} = G + \Delta$ where $\|\Delta\| \leq \varepsilon$
Proof - the basic idea

Take an almost isometry \( T : \mathcal{A}(B_n) \to \mathcal{B}; \| T \| \| T^{-1} \| < 1 + \epsilon_0 \)

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for some small \( \delta \) dependent only on \( \epsilon_0 \)

- show that any almost multiplicative functional \( G \) on \( \mathcal{A}(B_n) \) must be close to a multiplicative functional \( \tilde{G} = G + \Delta \) where \( \|\Delta\| \leq \epsilon \)

- show that the functional \( \tilde{G} \) may be selected in such a way that \( G \to \tilde{G} \) is a homeomorphism from the maximal ideal space \( \mathcal{M}(\mathcal{B}) \) of \( \mathcal{B} \) onto \( \overline{B}_n \)
Proof - the basic idea

Take an almost isometry $T : \mathcal{A}(B_n) \to \mathcal{B}; \|T\| \|T^{-1}\| < 1 + \varepsilon_0$

- show that $\mathcal{B}$ must automatically be a uniform algebra
- show that without loss of generality we may assume $T1 = 1$
- take a linear and multiplicative functional $F$ on $\mathcal{B}$ and define a functional on $\mathcal{A}(B_n)$ by $G = F \circ T$
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- show that any almost multiplicative functional $G$ on $\mathcal{A}(B_n)$ must be close to a multiplicative functional $\tilde{G} = G + \Delta$ where $\|\Delta\| \leq \varepsilon$

- show that the functional $\tilde{G}$ may be selected in such a way that $G \to \tilde{G}$ is a homeomorphism from the maximal ideal space $M(\mathcal{B})$ of $\mathcal{B}$ onto $\overline{B_n}$

- show that we can introduce an analytic structure on $M(\mathcal{B})$ such that all functions from $\mathcal{B}$ are analytic.
Theorem (KJ, 1985)

*Metric and algebraic deformations coincide for uniform algebras:*

Let $A$ be a complex uniform algebra then the following conditions are equivalent:

1. $k a b k \leq \varepsilon_1 k a k k b k$, for $a, b \in A$

2. $k a b k \leq \varepsilon_2 k a k k b k$, for $a, b \in A$

3. $k a b k \leq (1 + \varepsilon_3) k a k k b k$, for $a, b \in A$

4. $a b = T_1 (T a T b)$ where $T : A \rightarrow B$ is such that $k T k \leq 1 + \varepsilon_4$ it follows that $B$ is a uniform algebra and that the Chauquet boundaries of $A$ and $B$ are homeomorphic.
Theorem (KJ, 1985)

Metric and algebraic deformations coincide for uniform algebras: Let $A$ be a complex uniform algebra then the following conditions are equivalent

1. $\|a \cdot b - a \times b\| \leq \varepsilon_1 \|a\| \|b\|$, for $a, b \in A$

(we assume that all multiplications have the same unit).
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3. $\|a \times b\| \leq (1 + \varepsilon_3) \|a\| \|b\|$, for $a, b \in A$

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4. $a \times b = T^{-1}(Ta \cdot Tb)$ where $T : \mathcal{A} \to \mathcal{B}$ is such that $\|T\| \|T^{-1}\| \leq 1 + \varepsilon_4$

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Let $\mathcal{A}$ be a complex uniform algebra then the following conditions are equivalent

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it follows that $\mathcal{B}$ is a uniform algebra and that the Chauquet boundaries of $\mathcal{A}$ and $\mathcal{B}$ are homeomorphic.

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Almost multiplicative functionals

**Definition**

$F \in \mathcal{A}^*$ is $\delta$-multiplicative iff $|F(ab) - F(a)F(b)| \leq \delta \|a\| \|b\|$.

Examples

- $F = G + \Delta$ where $G$ is multiplicative and $\|\Delta\| \leq \varepsilon$,

- $F = G^T$ where $G$ is multiplicative and $\|G^T - I\| < 1 + \varepsilon$.

Problem

Must any almost multiplicative functional be near a multiplicative one?

**Definition**

A is functionally-stable or $f$-stable (or $AMNM$ algebra) if

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall F \in \mathcal{M}_\delta(A) \ \|F - G\| < \varepsilon,$

where $\mathcal{M}_\delta(A)$ is the set of $\delta$-multiplicative functionals on $A$.
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**Examples**

- \[ F = G + \Delta \text{ where } G \text{ is multiplicative and } \|\Delta\| \leq \varepsilon, \]
- \[ F = G \circ T \text{ where } G \text{ is multiplicative and } \|T\| \|T^{-1}\| < 1 + \varepsilon. \]
**Almost multiplicative functionals**

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\( \mathcal{A} \) is *functionally-stable* or *\( f \)-stable* (or AMNM algebra) if

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\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall F \in \mathcal{M}_\delta(\mathcal{A}) \ \exists G \in \mathcal{M}(\mathcal{A}) \ \|F - G\| \leq \varepsilon,
\]

where \( \mathcal{M}_\delta(\mathcal{A}) \) is the set of \( \delta \)–multiplicative functionals on \( \mathcal{A} \).
Almost multiplicative functionals

Example

$C(X)$ algebras are f-stable.
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Example (B. E. Johnson, 1986)

The convolution algebra $L^2(0,1)$ is $f$-stable, while $L^1(0,1)$ is not.
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Proof.

If $f_0 \in L^1(0, 1)$ is "near" the Dirac measure then $f \mapsto f \ast f_0$ is almost multiplicative.
## Almost multiplicative functionals

### Example

$C(X)$ algebras are f-stable.

### Theorem (B. E. Johnson, 1986)

*The disc algebra $A(D)$ is f-stable.*

### Example (B. E. Johnson, 1986)

The convolution algebra $L^2(0, 1)$ is f-stable, while $L^1(0, 1)$ is not.

### Proof.

If $f_0 \in L^1(0, 1)$ is "near" the Dirac measure then $f \longmapsto f \ast f_0$ is almost multiplicative.

### Example (S. J. Sidney, 1997)

There exists a non f-stable uniform algebra.
Theorem

- The ball algebras $A(B_n)$ are f-stable.
Theorem

- The ball algebras $A(B_n)$ are f-stable.
- Any uniform algebra with one generator is f-stable.
Almost multiplicative functionals - recent results

Theorem

- The ball algebras \( A(B_n) \) are f-stable.
- Any uniform algebra with one generator is f-stable.
- If \( K \subset \mathbb{C} \) is such that \( \mathbb{C} \setminus K \) has finitely many components and the closures of the components are disjoint then \( R(K) \) is f-stable.
Theorem

- The ball algebras $A(B_n)$ are f-stable.
- Any uniform algebra with one generator is f-stable.
- If $K \subset \mathbb{C}$ is such that $\mathbb{C}\setminus K$ has finitely many components and the closures of the components are disjoint then $R(K)$ is f-stable.

Theorem

Let $B(z) = \prod_{n=1}^{\infty} \frac{\bar{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \bar{\alpha}_nz}$ be a Blaschke product. Then

- if $B$ is an interpolating Blaschke product then $H^\infty / BH^\infty$ is f-stable;
Almost multiplicative functionals - recent results

**Theorem**

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Let $B(z) = \prod_{n=1}^{\infty} \frac{\tilde{\alpha}_n}{|\alpha_n|} \frac{\alpha_n - z}{1 - \tilde{\alpha}_nz}$ be a Blaschke product. Then

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Problem
Is $H^\infty(\mathbb{D})$ is f-stable? (Does $H^\infty(\mathbb{D})$ have an almost corona?)
Theorem

The ball algebras $\mathcal{A}(B_n)$ are $f$-stable.
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Step 1. Show that (without loss of generality) \( F \) may be represented by a probabilistic, nonatomic measure \( \mu_F \) on \( \partial B_n \).
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**Theorem**

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**Proof.** Let \( F \in \mathcal{M}_\delta (\mathcal{A}(B_n)) \). We show that \( F \) is near a multiplicative functional.

**Step 1.** Show that (without loss of generality) \( F \) may be represented by a probabilistic, nonatomic measure \( \mu_F \) on \( \partial B_n \).

**Step 2.** For \( w = (w_1, \ldots, w_n) \in B_n \) define \( \Phi_w : \bar{B}_n \to \bar{B}_n \) by

\[
\Phi_w(z) = \frac{w - P_z - \sqrt{1 - \|w\|^2} (z - P_z)}{1 - \langle z, w \rangle},
\]

where

\[
\langle z, w \rangle = \sum_{k=1}^n z_k \bar{w}_k \quad \text{and} \quad P_z = \frac{\langle z, w \rangle}{\|w\|^2} w.
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\Phi_{w_j}(z) \to w_0 \quad \text{pointwise on } \bar{B}_n \setminus \{w_0\}.
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Since $\mu_F$ has no atoms, $\phi$ extends to a continuous function on $\bar{B}_n$ such that $\phi(w) = w$ for $w \in \partial B_n$. 
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Since $\mu_F$ has no atoms, $\varphi$ extends to a continuous function on $\bar{B}_n$ such that $\varphi(w) = w$ for $w \in \partial B_n$. Hence $\varphi$ is surjective and there is a $w_0 \in B_n$ such that $\varphi(w_0) = 0$. 
Almost multiplicative functionals - proof

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$$F(z_k) = 0 \quad \text{for } k = 1, \ldots, n.$$
Step 4. Put $\mathcal{A}_0(B_n) \overset{df}{=} \{ f \in \mathcal{A}(B_n) : f(0) = 0 \}$ and define

$$T : (\mathcal{A}(B_n))^n \to \mathcal{A}_0(B_n) \quad \text{by} \quad T(f_1, \ldots, f_n) \overset{df}{=} \sum_{k=1}^{n} z_k f_k.$$
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$T$ is surjective and there are continuous linear selections $S_k$ [E. Stout]

$$\mathcal{A}_0(B_n) \ni \sum_{k=1}^{n} z_k f_k \overset{S_k}{\mapsto} f_k \in \mathcal{A}(B_n).$$

Hence $F\delta_0$.

As a special case, for $n=1$ we get Johnson’ s theorem.
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Problem

*For bounded pseudoconvex domains $\Omega$ in $\mathbb{C}^n$ describe small deformations of algebras $A(\Omega)$ and $H^\infty(\Omega)$ and characterize these domains for which the algebras are stable.*
Open problems - deformations of uniform algebras

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Is the property "$A$ has $n$ generators" for $n \in \mathbb{N} \cup \{\infty\}$ stable?
Problem

Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$, are $A(\Omega)$ and $H^\infty(\Omega)$ f-stable?

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Is any uniform algebra with two generators f-stable?

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Let $A$ be an f-stable uniform algebra. Is an ultrapower of $A$ f-stable? Is $l_\infty(A)$ f-stable?

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Let $B$ be a product of finitely many interpolating Blaschke products. Is $H^\infty/BH^\infty$ f-stable?
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Describe small deformations of topological algebras of analytic functions on bounded pseudoconvex domains $\Omega$ in $\mathbb{C}^n$
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Fact

For $n = 1$ there are partial results available, similar to Rochberg’s 1986 description of deformation of $\mathcal{A}(S)$ [M. Abel & KJ, 2003]
Open problems - deformation of non-commutative uniform algebras

Definition

\( \mathcal{A} \) is a uniform algebra iff \( \|a^2\| = \|a\|^2 \).
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**Problem**

*Describe small deformations of real uniform algebras.*
More open problems - separating maps

**Definition**

\( S : \mathcal{A} \rightarrow \mathcal{B} \) is separating iff \( f \cdot g = 0 \iff S(f) \cdot S(g) = 0 \).

**Theorem**

For compact \( X \), \( Y \) and a separating \( S \):

\( C(X) ! C(Y) \)

\( S \) may not be continuous (even if \( Y \) is a singleton), if \( S^{-1} \) exists then \( S \) is continuous and \( S^{-1} \) is separating.

**Problem**

Assume \( S : C(X) ! C(Y) \) is a separating bijection (\( X \), \( Y \) may not be compact), does it follow that \( S^{-1} \) is separating?

**Theorem (J. Araujo & KJ, 1999)**

Yes if \( X \subseteq \mathbb{R} \).
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THANK YOU!