Crossed products of pro-C*-algebras

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Basic Definitions

Definitions

1. A submultiplicative $*$-seminorm on a topological $*$-algebra A is a seminorm p on A which verifies the following relations:
   - $p(a^*) = p(a)$ for all $a \in A$;
   - $p(ab) \leq p(a)p(b)$ for all $a, b \in A$.
2. A locally m-convex $*$-algebra (shortly lmc*-algebra) is a Hausdorff topological $*$-algebra A whose topology is given by a directed family of submultiplicative $*$-seminorms $\{p_\lambda; \lambda \in \Lambda\}$.

Pro-C*-algebras

Definition

A seminorm p on a topological $*$-algebra A satisfies the C^*-condition (or is a C^*-seminorm) if

$$p\left(a^* a\right) = p\left(a\right)^2, \forall a \in A.$$

- It is known that such a seminorm must be submultiplicative and $*$-preserving.

Definition

A pro-C^*-algebra is a complete Hausdorff topological $*$-algebra A whose topology is given by a directed family of C^*-seminorms $\{p_\lambda; \lambda \in \Lambda\}$.

The terms used for pro-C^*-algebras:

- locally C^*-algebras (A. Inoue, M. Fragoulopoulou, A. Mallios, etc.)
- LMC^* -algebras (G. Lassner, K. Schmüdgen)
- b^*-algebras (C. Apostol)

The term pro-C^*-algebra was first used by D. Voiculescu and W. Arweson.

Let $A\left[\tau_{\Gamma}\right]$ and $B\left[\tau_{\Gamma'}\right]$ be two pro-C^*-algebras with $\Gamma = \{p_\lambda, \lambda \in \Lambda\}$ and $\Gamma' = \{q_\delta, \delta \in \Delta\}$.

Definitions

1. A linear map $\Phi : A\left[\tau_{\Gamma}\right] \to B\left[\tau_{\Gamma'}\right]$ is a morphism of pro-C^*-algebras if
   - $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in A\left[\tau_{\Gamma}\right]$;
   - $\Phi(a^*) = \Phi(a)^*$ for all $a \in A\left[\tau_{\Gamma}\right]$;
   - for each $\lambda \in \Lambda$, there is $\delta \in \Delta$ such that $p_\lambda(\Phi(a)) \leq q_\delta(a)$ for all $a \in A\left[\tau_{\Gamma}\right]$.
2. An isomorphism of pro-C^*-algebras is an invertible morphism of pro-C^*-algebras, Φ, such that Φ^{-1} is a morphism of pro-C^*-algebras.

Pro-C*-algebras (The Arens-Michael Decomposition)

Let $A\left[\tau_\Gamma\right]$ be a pro-C^*-algebra with the topology given by $\Gamma = \{p_\lambda; \lambda \in \Lambda\}$.

- For $\lambda \in \Lambda$, $\ker p_\lambda$ is a closed $*$-bilateral ideal and $A_\lambda = A/\ker p_\lambda$ is a C^*-algebra in the C^*-norm $\|\cdot\|_{p_\lambda}$ induced by p_λ (that is, $\|a\|_{p_\lambda} = p_\lambda(a), \forall a \in A$).
  - The canonical map from A to A_λ is denoted by π_λ^A ($\pi_\lambda^A(a) = a + \ker p_\lambda \forall a \in A$).
  - For $\lambda, \mu \in \Lambda$ with $\mu \leq \lambda$ there is a surjective C^*-morphism $\pi_{\lambda\mu}^A : A_\lambda \to A_\mu$ such that

$$\pi_{\lambda\mu}^A\left(a + \ker p_\lambda\right) = a + \ker p_\mu .$$

- $\{A_\lambda; \pi_{\lambda\mu}^A\}_{\lambda,\mu\in\Lambda}$ is an inverse system of C^*-algebras.

Theorem

Let $A\left[\tau_\Gamma\right]$ be a pro-C^*-algebra with $\tau_\Gamma = \{p_\lambda; \lambda \in \Lambda\}$. Then

$$A \equiv \lim_{\leftarrow} A_\lambda \text{ (Arens-Michael decomposition)}$$

up to an isomorphism of pro-C^*-algebras.

Pro-C*-algebras (cont.)

Definition

A morphism of pro-C^*-algebras $\Phi : A[\tau_\Gamma] \rightarrow A[\tau_\Gamma]$ is an inverse limit morphism if there is an inverse system $\{\varphi_\lambda\}_\lambda$ of C^*-morphisms such that $\Phi = \lim\limits_{\leftarrow} \varphi_\lambda$

Examples

- Any inverse limit of C^*-algebras is a pro-C^*-algebra.
- The product of C^*-algebras with the product topology is a pro-C^*-algebra.
- Let X be a countably compactly generated Hausdorff topological space (that is, $X = \lim\limits_{\rightarrow} K_n$ is a direct limit of a countable set of compact spaces). The $*$-algebra $C(X)$ of all continuous complex valued functions on X with the topology given by the family of C^*-seminorms $\{p_{K_n}\}_n$,
$$p_{K_n}(f) = \sup\{|f(x)|\,; x \in K_n\}$$
is a pro-C^*-algebra.
- The algebra $C_{cc}([0,1])$ (Michael)
The $*$-algebra $C([0,1])$ of all complex valued continuous functions on $[0,1]$ with the topology of uniform convergence on the countable compact subsets of $[0,1]$ is a pro-C^*-algebra which is not topologically isomorphic with any C^*-algebra.

Pro-C*-algebras (cont.)

Examples

- The algebra $L(H)$ (Inoue)
  Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be an inductive system of Hilbert spaces such that $\langle\cdot,\cdot\rangle_\lambda|_{H_\mu} = \langle\cdot,\cdot\rangle_\mu$ for all $\lambda,\mu\in\Lambda$ with $\mu\leq\lambda$.
  - $H = \lim\limits_{\rightarrow} H_\lambda$ with the inductive limit topology is called a locally Hilbert space.
  - $$L(H) = \{T : H \to H; T_\lambda = T|_{H_\lambda} \in L(H_\lambda) \text{ and } P_{\lambda\mu}T_\lambda = T_\lambda P_{\lambda\mu} \text{ for all } \lambda,\mu\in\Lambda \text{ with } \mu\leq\lambda\},$$
    where $P_{\lambda\mu}$ is the projection of H_λ on H_μ, is an algebra and $T \to T^*$ with $T^*|_{H_\lambda} = (T_\lambda)^*$ for all $\lambda\in\Lambda$ is an involution on $L(H)$.
  - For each $\lambda\in\Lambda$, the map $p_\lambda : L(H) \to [0,\infty)$ given by
    $$p_\lambda(T) = \left\| T|_{H_\lambda} \right\|_{L(H_\lambda)}$$
    is a C^*-seminorm on $L(H)$.
    $L(H)$ with the topology given by the family of C^*-seminorms $\{p_\lambda\}_{\lambda\in\Lambda}$ is a pro-C^*-algebra.

Pro-C*-algebras (cont.)

Examples

Moreover, for $\lambda, \mu \in \Lambda$ with $\mu \leq \lambda$,

$$\pi_{\lambda\mu}^{H}: L(H_{\lambda}) \to L(H_{\mu}), \pi_{\lambda\mu}^{H}(T) = i_{\lambda\mu}^{*} T i_{\lambda\mu},$$

where $i_{\lambda\mu}$ is the inclusion of H_{μ} into H_{λ}, is a C^{*}-morphism, $\{L(H_{\lambda}), \pi_{\lambda\mu}^{H}\}_{\lambda,\mu\in\Lambda}$ is an inverse system of C^{*}-algebras, and

$$L(H) \simeq \lim_{\leftarrow} L(H_{\lambda}).$$

The canonical maps are denoted by $\pi_{\lambda}^{H}, \lambda \in \Lambda$, $\pi_{\lambda}^{H}(T) = T|_{H_{\lambda}}$.

Theorem

(Inoue, 1971) Every pro-C^{*}-algebra A is isomorphic to a pro-C^{*}-subalgebra of $L(H)$ for some locally Hilbert space H.

Group actions on pro-C*-algebras

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra with the topology given by $\Gamma = \{p_\lambda; \lambda \in \Lambda\}$ and G a locally compact group.

$\text{Aut}(A[\tau_\Gamma]) = \{\varphi : A[\tau_\Gamma] \to A[\tau_\Gamma]; \varphi$ is an automorphism of pro-C^*-algebras$\}$

Definitions

1. An *action* of a G on $A[\tau_\Gamma]$ is a group morphism $\alpha : G \to \text{Aut}(A[\tau_\Gamma])$ such that the map $t \mapsto \alpha_t(a)$ from G to $A[\tau_\Gamma]$ is continuous for each $a \in A$.
2. An action α of G on $A[\tau_\Gamma]$ is *strongly bounded*, if for each $\lambda \in \Lambda$, there is $\mu \in \Lambda$ such that

$$p_\lambda(\alpha_t(a)) \leq p_\mu(a), \forall\ t \in G, \forall\ a \in A.$$

3. An action α is *an inverse limit action*, if $\Gamma(A, G)$ is a cofinal subset of Γ, $\Gamma(A, G) = \{p_\lambda \in \Gamma; p_\lambda(\alpha_t(a)) = p_\lambda(a), \forall a \in A, \forall t \in G\}$.

Remark

If α is an inverse limit action, we suppose that $\Gamma(A, G) = \Gamma$. Then, for each $t \in G$, $\alpha_t = \varprojlim \alpha_t^\lambda$, where $\alpha^\lambda, \lambda \in \Lambda$, are actions of G on A_λ.

Remark

1. *If A is a C^*-algebra, then any action of G on $A[\tau_\Gamma]$ is strongly bounded.*
2. *Any inverse limit action of G on $A[\tau_\Gamma]$ is strongly bounded.*
3. *If G is compact, then any action of G on $A[\tau_\Gamma]$ is an inverse limit action.*

Examples

- Let (G, X) be a transformation group with $X = \lim\limits_{n\to} K_n$ a countably compactly generated Hausdorff topological space with the property that

$$\forall\ n \in \mathbb{N}, \exists\ m \in \mathbb{N} \text{ s.t. } G \cdot K_n \subseteq K_m.$$

Then there is an action α of G on the pro-C^*-algebra $C(X)$, given by

$$\alpha_t(f)(x) = f\left(t^{-1} \cdot x\right), \forall f \in C(X), \forall t \in G, \forall x \in X.$$

Moreover, this action is *strongly bounded,* since for each n there is m such that

$$\begin{aligned} p_{K_n}(\alpha_t(f)) &= \sup\{\left|f\left(t^{-1} \cdot x\right)\right|; x \in K_n\} \\ &\leq \sup\{|f(y)|; y \in K_m\} = p_{K_m}(f) \end{aligned}$$

for all $f \in C(X)$, for all $t \in G$.

Examples

- Let $X = \lim_{n\to} K_n$ be a compactly countably generated Hausdorff topological space and h a homeomorphism from X onto X with the property that

$$\forall\ n \in \mathbb{N}, \exists\ m \in \mathbb{N} \text{ s.t. } h^k(K_n) \subseteq K_m, \forall k \in \mathbb{Z}.$$

  Then the map $n \to \alpha_n$ from $\mathbb{Z}$ to $\mathrm{Aut}(C(X))$, where $\alpha_n(f) = f \circ h^n$, is a strong bounded action of $\mathbb{Z}$ to $C(X)$.
  If $h(K_n) = K_n$ for each positive integer n, then α is an inverse limit action.
- The map $\alpha : C_{cc}[0,1] \to C_{cc}[0,1])$ given by

$$\alpha(f)(x) = f(1-x), \forall f \in C_{cc}[0,1], \forall x \in [0,1]$$

  is a pro-C^*-automorphism. It is not an inverse limit automorphism. The action of $\mathbb{Z}$ on $C_{cc}[0,1]$ induced by α, $n \mapsto \alpha^n$ is a strong bounded action which is not an inverse limit action.

The covariance algebra

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra with the topology given by $\Gamma = \{p_\lambda; \lambda \in \Lambda\}$, G a locally compact group and α an action of G on A.

- $C_c(G, A[\tau_\Gamma]) = \{f : G \to A[\tau_\Gamma]; f$ is continuous with compact support$\}$ is a $*$-algebra with:
  - $(f * h)(t) = \int_G f(g)\alpha_g\left(h(g^{-1}t)\right) dg$
  - $f^\#(t) = \Delta\left(t^{-1}\right)\alpha_t\left(f\left(t^{-1}\right)\right)^*$, where Δ is the modular function on G.
- For each λ, consider the seminorm N_λ on $C_c(G, A[\tau_\Gamma])$ given by

$$N_\lambda(f) = \int_G p_\lambda\left(f(g)\right) dg.$$

The covariance algebra (cont.)

- Suppose that α is strongly bounded. Then, for each λ, there is μ such that

$$\begin{aligned} N_\lambda(f*h) &= \int_G p_\lambda\left(\left(f*h\right)(t)\right)dt \\ &= \int_G p_\lambda\Big(\int_G f(g)\alpha_g\left(h(g^{-1}t)\right)dg\Big)dt \\ &\leq \int_G\int_G p_\lambda\left(f(g)\alpha_g\left(h(g^{-1}t)\right)\right)dgdt \\ &\leq \int_G\int_G p_\mu\left(f(g)\right)p_\mu\left(h(g^{-1}t)\right)dtdg \\ &= N_\mu(f)N_\mu(h),\ \forall f,h\in C_c\left(G,A\left[\tau_\Gamma\right]\right) \end{aligned}$$

  and

$$N_\lambda(f^{\#})\leq N_\mu(f),\forall f\in C_c\left(G,A\left[\tau_\Gamma\right]\right).$$

- $C_c\left(G,A\left[\tau_\Gamma\right]\right)$ is a locally convex $*$-algebra with the topology given by the family of seminorms $\{N_\lambda\}_{\lambda\in\Lambda}$.

The covariance algebra (cont.)

Definition

The covariance algebra $L^1(G, \alpha, A[\tau_\Gamma])$ is the Hausdorff completion of $C_c(G, A[\tau_\Gamma])$.

The enveloping pro-C*-algebra of an lmc *-algebra

Let $A\left[\tau_\Gamma\right]$ be an lmc *-algebra with bounded approximate unit and the topology given by $\{p_\lambda\}_{\lambda\in\Lambda}$.

Definition

A $*$-representation of $A\left[\tau_\Gamma\right]$ on a Hilbert space H is a continuous $*$-morphism $\varphi : A\left[\tau_\Gamma\right] \rightarrow L(H)$.

- $I = \left\{a \in A; \varphi\left(a\right) = 0, \forall *\text{-representation } \varphi \text{ of } A\left[\tau_\Gamma\right]\right\}$ is a closed $*$-bilateral ideal of $A\left[\tau_\Gamma\right]$.
  - A/I is an algebra with involution.
  - For each $\lambda, \widehat{p_\lambda} : A/I \rightarrow [0, \infty)$, $\widehat{p_\lambda}\left(a + I\right) = \sup\{\left\|\varphi\left(a\right)\right\|; \left\|\varphi\left(a\right)\right\| \leq p_\lambda\left(a\right), \forall a \in A\}$ is a C^*-seminorm on A/I.

Definition

The enveloping pro-C^*-algebra of $A\left[\tau_\Gamma\right]$ is the pro-C^*-algebra obtained by the completion of A/I with respect to the topology given by the family of C^*-seminorms $\{\widehat{p_\lambda}\}_{\lambda\in\Lambda}$.

The case of inverse limit actions

We suppose that α is an inverse limit action.
Then $\{N_\lambda\}_{\lambda\in\Lambda}$ is a family of submultiplicative $*$-seminorms and $L^1(G,\alpha,A[\tau_\Gamma])$ is a complete locally m-convex $*$-algebra with bounded approximate unit.

Definition

(MJ)The full crossed product of $A[\tau_\Gamma]$ by α is the the enveloping pro-C^*-algebra of $L^1(G,\alpha,A[\tau_\Gamma])$, denoted by $G\times_\alpha A[\tau_\Gamma]$.

- $\alpha_t = \lim\limits_{\leftarrow}\alpha_t^\lambda$, where $\alpha^\lambda, \lambda\in\Lambda$, are actions of G on A_λ
  - For $\mu\leq\lambda$, there is a C^*-mporphism $\pi_{\lambda\mu}: G\times_{\alpha^\lambda} A_\lambda \to G\times_{\alpha^\mu} A_\mu$ such that $\pi_{\lambda\mu}(f) = \pi_{\lambda\mu}^A\circ f$ for all $f\in C_c(G,A_\lambda)$
  - $\{G\times_{\alpha^\lambda} A_\lambda, \pi_{\lambda\mu}\}_{\lambda,\mu\in\Lambda}$ is in inverse system of C^*-algebras
  - $\lim\limits_{\leftarrow} G\times_{\alpha^\lambda} A_\lambda$ is the full crossed product of $A[\tau_\Gamma]$ by α (Phillips, 1989)

Proposition

The pro-C^*-algebras $G\times_\alpha A[\tau_\Gamma]$ and $\lim G\times_{\alpha^\lambda} A_\lambda$ are isomorphic.

The case general case

We suppose that α is not an inverse limit action.
In this case, the seminorms N_λ, $\lambda \in \Lambda$ are not submultiplicative, and so $L^1(G, \alpha, A[\tau_\Gamma])$ is not a locally m-convex $*$-algebra.

Multiplier algebras

- Let $A[\tau_\Gamma]$ be a pro-C^*-algebra. A multiplier of $A[\tau_\Gamma]$ is a pair of (l, r) of linear maps $l, r : A[\tau_\Gamma] \to A[\tau_\Gamma]$ such that are respectively left and right $A[\tau_\Gamma]$-module homomorphisms and $r(a)b = al(b)$ for all $a, b \in A[\tau_\Gamma]$.
- $M(A[\tau_\Gamma]) = \{(l, r); (l, r)$ is a multiplier of $A[\tau_\Gamma]\}$ is a pro-C^*-algebra with:
  - $(l_1, r_1)(l_2, r_2) = (l_1 l_2, r_2 r_1)$;
  - $(l, r)^* = (r^*, l^*)$, where $r^*(a) = r(a^*)^*$ and $l^*(a) = l(a^*)^*$ for all $a \in A[\tau_\Gamma]$;
  - the topology given by the family of C^*-seminorms $\{\widetilde{p}_\lambda\}_{\lambda \in \Lambda}$, $\widetilde{p}_\lambda(l, r) = \sup\{p_\lambda(l(a)); p_\lambda(a) \leq 1\}$.
- The strict topology on $M(A[\tau_\Gamma])$ is given by the family of seminorms $\{p_{\lambda,a}\}_{(\lambda,a)\in\Lambda\times A}$, $p_{\lambda,a}(l, r) = l(a) + r(a)$.

Covariant pro-C*-morphisms

Let $A\left[\tau_{\Gamma}\right]$ and $B\left[\tau_{\Gamma}^{\prime}\right]$ be a pro-C^{*}-algebras and α an action of a G on $A\left[\tau_{\Gamma}\right]$.

Definitions

1. A pro-C^{*}-morphism $\varphi : A\left[\tau_{\Gamma}\right] \rightarrow M(B\left[\tau_{\Gamma^{\prime}}\right])$ is non-degenerate if $\left[\varphi\left(A\left[\tau_{\Gamma}\right]\right) B\left[\tau_{\Gamma^{\prime}}\right]\right] = B\left[\tau_{\Gamma^{\prime}}\right]$.
2. A covariant pro-C^{*}-morphism from $A\left[\tau_{\Gamma}\right]$ to $B\left[\tau_{\Gamma}^{\prime}\right]$ is a pair (φ, u) consisting of a pro-C^{*}-morphism $\varphi : A\left[\tau_{\Gamma}\right] \rightarrow M(B\left[\tau_{\Gamma^{\prime}}\right])$ and a strictly continuous morphism $u : G \rightarrow \mathcal{U}\left(M(B\left[\tau_{\Gamma^{\prime}}\right])\right)$ such that
$$\varphi\left(\alpha_{t}\left(a\right)\right) = u_{t}\varphi\left(a\right) u_{t^{-1}}, \forall a \in A\left[\tau_{\Gamma}\right], \forall t \in G.$$
3. A covariant pro-C^{*}-morphism (φ, u) is non-degenerate if φ is non-degenerate.

Proposition

If $\varphi : A\left[\tau_{\Gamma}\right] \rightarrow M(B\left[\tau_{\Gamma^{\prime}}\right])$ is a non-degenerate pro-C^{*}-morphism, then there is a unique pro-C^{*}-morphism $\overline{\varphi} : M\left(A\left[\tau_{\Gamma}\right]\right) \rightarrow M(B\left[\tau_{\Gamma^{\prime}}\right])$ such that $\overline{\varphi}\left(a\right) = \varphi\left(a\right)$ for all $a \in A\left[\tau_{\Gamma}\right]$.

Covariant representations

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra with the topology given by $\Gamma = \{p_\lambda; \lambda \in \Lambda\}$, G a locally compact group and α an action of G on A.

Definition

A non-degenerate covariant representation of $A[\tau_\Gamma]$ is a triple (φ, u, H), where (φ, H) is a non-degenerate $*$-representation of $A[\tau_\Gamma]$ and (u, H) is a unitary representation of G such that

$$\varphi(\alpha_t(a)) = u_t \varphi(a) u_{t^{-1}}, \forall a \in A[\tau_\Gamma], \forall t \in G.$$

Proposition

If α is strongly bounded, then there is a non-degenerate covariant $*$-representation of $A[\tau_\Gamma]$.

Proof.

Let $(\varphi_\lambda, \mathcal{H})$ be a non-degenerate $*$-representation of A_λ. Then $(\varphi, \mathcal{H})$, where $\varphi = \varphi_\lambda \circ \pi_\lambda^A$, is a non-degenerate $*$-representation of $A[\tau_\Gamma]$. □

Covariant representations

Proof.

α is strongly bounded $\Rightarrow \exists\ p_\mu \in \Gamma$ s.t. $p_\lambda\left(\alpha_s\left(a\right)\right) \leq p_\mu\left(a\right)$, $\forall\ s \in G$, $\forall$ $a \in A\left[\tau_\Gamma\right]$.

From

$$\int_G \left\|\varphi\left(\alpha_{s^{-1}}\left(a\right)\right)\left(\xi\left(s\right)\right)\right\|^2 ds \leq \int_G \left\|\varphi\left(\alpha_{s^{-1}}\left(a\right)\right)\right\|^2 \left\|\xi\left(s\right)\right\|^2 ds$$

$$\leq \int_G p_\lambda\left(\alpha_{s^{-1}}\left(a\right)\right)^2 \left\|\xi\left(s\right)\right\|^2 ds \leq p_\mu\left(a\right)^2 \left\|\xi\right\|^2\ \forall a \in A,\ \forall\ \xi \in L^2(G, H)$$

$\Rightarrow$ the map $s \mapsto \varphi\left(\alpha_{s^{-1}}\left(a\right)\right)\left(\xi\left(s\right)\right)$ defines an element in $L^2(G, \mathcal{H})$

$\Rightarrow \exists\ \widetilde{\varphi}\left(a\right) \in L(L^2(G, \mathcal{H}))$ s.t $\widetilde{\varphi}\left(a\right)\left(\xi\right)\left(s\right) = \varphi\left(\alpha_{s^{-1}}\left(a\right)\right)\left(\xi\left(s\right)\right)$, $\forall\ \xi \in L^2(G, \mathcal{H}), \forall s \in G$

$\Rightarrow \exists\ \widetilde{\varphi} : A\left[\tau_\Gamma\right] \to L(L^2(G, \mathcal{H}))$ a non-degenerate $*$-representation of $A\left[\tau_\Gamma\right]$.

If $\lambda_G^{\mathcal{H}}$ be the unitary $*$-representation of G on $L^2\left(G, \mathcal{H}\right)$ given by

$$\left(\lambda_G^{\mathcal{H}}\right)_t\left(\xi\right)\left(s\right) = \xi\left(t^{-1}s\right),$$

then $\left(\widetilde{\varphi}, \lambda_G^{\mathcal{H}}, L^2\left(G, \mathcal{H}\right)\right)$ is a non-degenerate covariant $*$-representation of $A\left[\tau_\Gamma\right]$. □

The full crossed product

Definition

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra, G a locally compact group and α an action of G on A. The full crossed product of $A[\tau_\Gamma]$ by α is a pro-C^*-algebra, denoted by $G \times_\alpha A[\tau_\Gamma]$, together with a non-degenerate covariant pro-C^*-morphism (i_A, i_G) such that

1. $\overline{\text{span}}\{i_A(a)\, i_G(f)\, ; a \in A, f \in C_c(G)\} = G \times_\alpha A[\tau_\Gamma]$;
2. for each non-degenerate covariant $*$-representation $(\varphi, u, \mathcal{H})$ of $A[\tau_\Gamma]$, there is a unique non-degenerate $*$-representation $(\Phi, \mathcal{H})$ of $G \times_\alpha A[\tau_\Gamma]$ such that $\overline{\Phi} \circ i_A = \varphi$ and $\overline{\Phi} \circ i_G = u$.

The full crossed product

Proposition

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra, G a locally compact group, and α an action of G on $A[\tau_\Gamma]$ such that there is the full crossed product of $A[\tau_\Gamma]$ by α. If (φ, u) is a non-degenerate covariant pro-C^*-morphism from $A[\tau_\Gamma]$ to a pro-C^*-algebra $B[\tau_{\Gamma'}]$, then there is a unique non-degenerate pro-C^*-morphism $\varphi \times u : G \times_\alpha A[\tau_\Gamma] \to M(B[\tau_{\Gamma'}])$ such that

$$\overline{\varphi \times u} \circ i_A = \varphi \text{ and } \overline{\varphi \times u} \circ i_G = u.$$

The full crossed product

Corollary

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra, G a locally compact group and α an action of G on A. If there is the full crossed product of $A[\tau_\Gamma]$ by α, then it is unique up to an isomorphism of pro-C^*-algebras.

Corollary

Let $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ be two pro-C^*-algebras, G a locally compact group, α and β actions of G on $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ such that there are the full crossed product of $A[\tau_\Gamma]$ by α and the full crossed product of $B[\tau_{\Gamma'}]$ by β. If there is a pro-C^*-isomorphism $\varphi : A[\tau_\Gamma] \to B[\tau_{\Gamma'}]$ such that

$$\varphi \circ \alpha_t = \beta_t \circ \varphi, \forall t \in G,$$

then the pro-C^*-algebras $G \times_\alpha A[\tau_\Gamma]$ and $G \times_\beta B[\tau_{\Gamma'}]$ are isomorphic.

The full crossed product

Proposition

If α is an inverse limit action of a locally compact group G on a pro-C^*-algebra $A[\tau_\Gamma]$, then there is the full crossed product of $A[\tau_\Gamma]$ by α and it is isomorphic to the enveloping pro-C^*-algebra of the covariance algebra.

The full crossed product

Proposition

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra, G a locally compact group and α an action of G on $A[\tau_\Gamma]$ such that there is the full crossed product of $A[\tau_\Gamma]$ by α. There is a bijective correspondence between non-degenerate covariant $*$-representations of $A[\tau_\Gamma]$ and non-degenerate $*$-representations of $G \times_\alpha A[\tau_\Gamma]$.

Main theorem

We show that if the action α of a locally compact group G on a pro-C^*-algebra $A[\tau_\Gamma]$ is strongly bounded, then there is the full crossed product of $A[\tau_\Gamma]$ by α.

Theorem

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra with the topology given by $\Gamma = \{p_\lambda; \lambda \in \Lambda\}$, G a locally compact group and α a strong bounded action of G on $A[\tau_\Gamma]$. Then there is a locally Hilbert space $\mathcal{H}$ and a covariant pro-C^*-morphism (j_A, j_G) of $A[\tau_\Gamma]$ on $L(\mathcal{H})$ such that

1. $\overline{span}\{j_A(a)\, j_G(f)\,; a \in A, f \in C_c(G)\} = B$ *is a pro-C^*-subalgebra of* $L(\mathcal{H})$;
2. $j_A(a) \in M(B)$ *for all* $a \in A$ *and* $j_G(t) \in M(B)$ *for all* $t \in G$;
3. *for each* $(\varphi, u, \mathcal{K})$ *non-degenerate covariant $*$-representation of* $A[\tau_\Gamma]$, *there is a unique non-degenerate $*$-representation* $(\Phi, \mathcal{K})$ *of* B *such that* $\overline{\Phi} \circ j_A = \varphi$ *and* $\overline{\Phi} \circ j_G = u$.

Proof (sketch)

Proof.

For each $\lambda \in \Lambda$, let $S_\lambda = \{(\varphi, u, \mathcal{K}) ; (\varphi, u, \mathcal{K})$ is a non-degenerate covariant $*$-representation of $A[\tau_\Gamma]$ such that $\|\varphi(a)\| \leq p_\lambda(a)$ for all $a \in A\}$ and $\left(\varphi^\lambda, u^\lambda, \mathcal{K}_\lambda\right) = \bigoplus_{S_\lambda} (\varphi, u, \mathcal{K})$.

- $\mathcal{H} = \lim\limits_{\rightarrow} \mathcal{H}_\lambda$, where $\mathcal{H}_\lambda = \bigoplus\limits_{\mu \leq \lambda} \mathcal{K}_\mu$
- $j_A(a) \bigoplus\limits_{\mu \leq \lambda} \xi_\mu = j_A^\lambda(a) \bigoplus\limits_{\mu \leq \lambda} \xi_\mu = \bigoplus\limits_{\mu \leq \lambda} \varphi^\mu(a) \xi_\mu$, $a \in A$
- $j_G(t) \bigoplus\limits_{\mu \leq \lambda} \xi_\mu = j_G^\lambda(t) \bigoplus\limits_{\mu \leq \lambda} \xi_\mu = \bigoplus\limits_{\mu \leq \lambda} u^\mu(t) \xi_\mu$, $t \in G$

□

Proof (cont.)

Proof.

(1) To show that B is a pro-C^*-subalgebra, we must show that B is closed under taking multiplication and adjoints. For this, it is sufficient to show that for each $\lambda \in \Lambda$,

$$\begin{aligned} \pi_\lambda^{\mathcal{H}}(j_G(f)\, j_A(a)),\, \pi_\lambda^{\mathcal{H}}(j_A(b)\, j_G(h)\, j_A(a)\, j_G(f)) &\in \pi_\lambda^{\mathcal{H}}(B), \\ \forall a, b &\in A, \forall f, h \in C_c(G). \end{aligned}$$

Since the map $s \mapsto \pi_\lambda^A(f(s)\, \alpha_s(a))$ defines an element in $C_c(G, A_\lambda)$, there is a net $\{\pi_\lambda^A(a_i) \otimes f_i\}_i$ in $A_\lambda \otimes_{\text{alg}} C_c(G)$, which converges uniformly to this map.

We have:

$$\pi_\lambda^{\mathcal{H}}(j_G(f)\, j_A(a)) = j_G^\lambda(f)\, \pi_\lambda^{\mathcal{H}}(j_A(a)) = \int_G f(s)\, j_G^\lambda(s)\, \pi_\lambda^{\mathcal{H}}(j_A(a))\, ds$$

$$= \int_G f(s)\, \pi_\lambda^{\mathcal{H}}(j_G(s)\, j_A(a))\, ds = \int_G f(s)\, \pi_\lambda^{\mathcal{H}}(j_A(\alpha_s(a))\, j_G(s))\, ds \qquad \square$$

Proof (cont.)

Proof.

and

$$\pi_\lambda^{\mathcal{H}}\left(j_A\left(a_i\right)j_G\left(f_i\right)\right) = \pi_\lambda^{\mathcal{H}}\left(j_A\left(a_i\right)\right)j_G^\lambda\left(f_i\right) = \int_G f_i(s)\pi_\lambda^{\mathcal{H}}\left(j_A\left(a_i\right)j_G\left(s\right)\right)ds.$$

Then

$$\begin{aligned}&\left\|\pi_\lambda^{\mathcal{H}}\left(j_G\left(f\right)j_A\left(a\right)\right) - \pi_\lambda^{\mathcal{H}}\left(j_A\left(a_i\right)j_G\left(f_i\right)\right)\right\| \\ &\leq \int_G \left\|\pi_\lambda^{\mathcal{H}}\left(f\left(s\right)j_A\left(\alpha_s\left(a\right)\right)j_G\left(s\right) - f_i(s)j_A\left(a_i\right)j_G\left(s\right)\right)\right\| ds \\ &\leq \int_G \left\|\pi_\lambda^{\mathcal{H}}\left(j_A\left(f\left(s\right)\alpha_s\left(a\right) - f_i(s)a_i\right)\right)\right\| ds \\ &\leq \int_G p_\lambda\left(f\left(s\right)\alpha_s\left(a\right) - f_i(s)a_i\right) ds \\ &\leq M \sup\{\left\|\pi_\lambda^A\left(f\left(s\right)\alpha_s\left(a\right)\right) - \pi_\lambda^A\left(f_i(s)a_i\right)\right\|; s \in K\}.\end{aligned}$$

Therefore, $\pi_\lambda^{\mathcal{H}}\left(j_G\left(f\right)j_A\left(a\right)\right) \in \pi_\lambda^{\mathcal{H}}\left(B\right)$.
In the same manner, we show that
$\pi_\lambda^{\mathcal{H}}\left(j_A\left(b\right)j_G\left(h\right)j_A\left(a\right)j_G\left(f\right)\right) \in \pi_\lambda^{\mathcal{H}}\left(B\right)$. □

Proof (cont.)

Proof.

(2) To show that $j_A(a), j_G(t) \in M(B)$, it is sufficient to show that

$$\pi_\lambda^{\mathcal{H}}(j_A(a) j_A(b) j_G(f)), \pi_\lambda^{\mathcal{H}}(j_A(b) j_G(f) j_A(a)) \in \pi_\lambda^{\mathcal{H}}(B)$$

and

$$\pi_\lambda^{\mathcal{H}}(j_G(t) j_A(b) j_G(f)), \pi_\lambda^{\mathcal{H}}(j_A(b) j_G(f) j_G(t)) \in \pi_\lambda^{\mathcal{H}}(B)$$

$$\forall\ b \in A, \forall\ f \in C_c(G), \forall \lambda \in \Lambda.$$

(3) Let $(\psi, v, H_{\psi,v})$ be a non-degenerate covariant $*$-representation of $A[\tau_\Gamma]$. Then there are and $\lambda \in \Lambda$ and $(\varphi, u, \mathcal{K}) \in S_\lambda$ such that $(\psi, v, H_{\psi,v})$ and $(\varphi, u, \mathcal{K})$ are unitarily equivalent.

Consider the map $\Phi : L(H) \to L(H_{\psi,v})$,

$$\Phi(T) = UPQ\pi_\lambda^{\mathcal{H}}(T) Q^* P^* U^*,$$

where Q is the projection of $\mathcal{H}_\lambda$ on $\mathcal{K}_\lambda$, P is the projection of $\mathcal{K}_\lambda$ on $\mathcal{K}$ and U is the unitary operator from $\mathcal{K}$ to $H_{\psi,v}$.

Then $\Psi : B \to L(H_{\psi,v})$, $\Psi(b) = \Phi(b)$ is a non-degenerate $*$-representation such that $\Psi \circ j_A = \psi$ and $\Psi \circ j_G = v$. □

The reduced crossed product

Let α be a strongly bounded action of a locally compact group G on a pro-C^*-algebra $A[\tau_\Gamma]$.
For each $a \in A$, the map

$$t \mapsto \alpha_{t^{-1}}(a)$$

defines an element in $C_b(G, A[\tau_\Gamma]) \subseteq M(A[\tau_\Gamma] \otimes C_0(G))$.
We can define a map $\widetilde{\alpha} : A[\tau_\Gamma] \to M(A[\tau_\Gamma] \otimes C_0(G))$ such that

$$\widetilde{\alpha}(a)(t) = \alpha_{t^{-1}}(a).$$

Lemma

$\widetilde{\alpha}$ is a non-degenerate faithful pro-C^*-morphism from $A[\tau_\Gamma]$ to $M(A[\tau_\Gamma] \otimes C_0(G))$.

The reduced crossed product

Let $\varphi : A[\tau_\Gamma] \to M(B[\tau_{\Gamma'}])$ be a non-degenerate pro-C^*-morphism and let $M : C_0(G) \to L(L^2(G))$ be the representation by multiplication operators.
Then $\varphi \otimes M : A[\tau_\Gamma] \otimes C_0(G) \to M(B[\tau_{\Gamma'}] \otimes \mathcal{K}(L^2(G)))$
is a non-degenerate pro-C^*-morphism.
Let $\lambda_G \colon G \to \mathcal{U}(L^2(G))$ be the left unitary representation of G on $L^2(G)$ given by $(\lambda_G)_t(\xi)(s) = \xi(t^{-1}s)$.

Lemma

The pair $(\widetilde{\varphi}, 1 \otimes \lambda_G)$, where $\widetilde{\varphi} = \overline{\varphi \otimes M} \circ \widetilde{\alpha}$, is a non-degenerate covariant pro-C^*-morphism from $A[\tau_\Gamma]$ to $B[\tau_{\Gamma'}] \otimes \mathcal{K}(L^2(G))$.

We have see that there is a unique non-degenerate pro-C^*-morphism $\widetilde{\varphi} \times (1 \otimes \lambda_G) \colon G \times_\alpha A[\tau_\Gamma] \to M\left(B[\tau_{\Gamma'}] \otimes \mathcal{K}(L^2(G))\right)$ such that $\widetilde{\varphi} \times (1 \otimes \lambda_G) \circ i_A = \widetilde{\varphi}$ and $\widetilde{\varphi} \times (1 \otimes \lambda_G) \circ i_G = 1 \otimes \lambda_G$.
If $\varphi = \mathrm{id}_A$, the non-degenerate pro-C^*-morphism $\widetilde{\mathrm{id}}_A \times (1 \otimes \lambda_G)$ is denoted by Λ_A^G.

The reduced crossed product

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra, G a locally compact group, α a strong bounded action of G on $A[\tau_\Gamma]$.

Definition

The reduced crossed product of $A[\tau_\Gamma]$ by α is the pro-C^*-subalgebra $G\times_{\alpha,r} A[\tau_\Gamma]$ of $M(A[\tau_\Gamma]\otimes\mathcal{K}(L^2(G))$ generated by the range of Λ_A^G.

Proposition

If G is amenable then the pro-C^*-morphism $\Lambda_A^G : G\times_\alpha A[\tau_\Gamma]\to M\left(A[\tau_\Gamma]\otimes\mathcal{K}(L^2(G))\right)$ is injective.

Question

Suppose that G is amenable and α is a strong bounded action of G on a pro-C^*-algebra $A[\tau_\Gamma]$. Are isomorphic pro-C^*-algebras $G\times_\alpha A[\tau_\Gamma]$ and $G\times_{\alpha,r} A[\tau_\Gamma]$?

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Thank you for your attention!