Ordered algebras with an order unit

Jukka Kauppi

University of Oulu

Weighted function algebras

Weighted function algebras

Let $v\colon X \to \mathbb{R}$ be a continuous real-valued function on a locally compact Hausdorff space X such that $\inf_{t\in X} v(t) > 0$. We define

$$C_b^v(X) := \{f \in C(X) : \ vf \text{ is bounded on } X\},$$

$$C_0^v(X) := \{f \in C(X) : \ vf \text{ vanishes at infinity on } X\}.$$

Weighted function algebras

Let $v\colon X \to \mathbb{R}$ be a continuous real-valued function on a locally compact Hausdorff space X such that $\inf_{t\in X} v(t) > 0$. We define

$$C_b^v(X) := \{f \in C(X) : \ vf \text{ is bounded on } X\},$$

$$C_0^v(X) := \{f \in C(X) : \ vf \text{ vanishes at infinity on } X\}.$$

When equipped with the pointwise operations and the *weighted supremum norm*

$$\|f\|_v := \sup_{t\in X} v(t)|f(t)|,$$

these sets become real Banach algebras.

Ordered algebras

Ordered algebras

In this talk, the term *algebra* will mean a non-trivial real vector space A equipped with a bilinear map $(a, b) \mapsto ab$ from $A \times A$ into A.

Ordered algebras

In this talk, the term *algebra* will mean a non-trivial real vector space A equipped with a bilinear map $(a, b) \mapsto ab$ from $A \times A$ into A.

Definition

An *ordered algebra* is a pair (A, A_+), where A is an algebra and A_+ is a non-empty subset of A satisfying the following three conditions:

(i) $rA_+ + sA_+ \subseteq A_+$ for all $r, s \in \mathbb{R}_+$;

(ii) $A_+A_+ \subseteq A_+$;

(iii) $A_+ \cap -A_+ = \{0\}$.

The set A_+ is called the *positive cone* of A.

Remark
Let (A, A_+) be an ordered algebra. For each pair $a, b \in A$, we write

$$a \leq b \quad \text{if} \quad b - a \in A_+.$$

Then $\leq$ defines a partial ordering on A which is compatible with the algebraic structure.

Remark

Let (A, A_+) be an ordered algebra. For each pair $a, b \in A$, we write

$$a \leq b \quad \text{if} \quad b - a \in A_+.$$

Then $\leq$ defines a partial ordering on A which is compatible with the algebraic structure.

Example

For a subalgebra B of the algebra $C(Y)$ of all continuous real-valued functions on an arbitrary topological space Y, put

$$B_+ := \{f \in B : \ f(t) \in \mathbb{R}_+ \text{ for all } t \in Y\}.$$

Then (B, B_+) is an ordered algebra.

Order unit

Definition

Let (A, A_+) be an ordered algebra. An *order unit* for A is an element $u \in A_+$ with the following two properties:

(i) each $a \in A$ satisfies $ru - a \in A_+$ and $ru + a \in A_+$ for some constant $r \in \mathbb{R}_+$;

(ii) each $a \in A$ with $ru + a \in A_+$ for all $r \in \mathbb{R}_+ \setminus \{0\}$ satisfies $a \in A_+$.

Definition

Let (A, A_+) be an ordered algebra. An *order unit* for A is an element $u \in A_+$ with the following two properties:

(i) each $a \in A$ satisfies $ru - a \in A_+$ and $ru + a \in A_+$ for some constant $r \in \mathbb{R}_+$;

(ii) each $a \in A$ with $ru + a \in A_+$ for all $r \in \mathbb{R}_+ \setminus \{0\}$ satisfies $a \in A_+$.

Example

Let B be a subalgebra of $C_b^v(X)$ containing the function $1/v$. Then (B, B_+), where B_+ is defined as above, is an ordered algebra with order unit $1/v$.

Algebraic order unit

Definition

Let (A, A_+) be an ordered algebra. By an *algebraic order unit* for A, we mean an order unit u for A with the following two additional properties:

(i) each pair $a, b \in A$ satisfies $u(ab) = (ua)b$ and $(ab)u = a(bu)$;

(ii) each $a \in A$ with $uau \in A_+$ satisfies $a \in A_+$.

Definition

Let (A, A_+) be an ordered algebra. By an *algebraic order unit* for A, we mean an order unit u for A with the following two additional properties:

(i) each pair $a, b \in A$ satisfies $u(ab) = (ua)b$ and $(ab)u = a(bu)$;
(ii) each $a \in A$ with $uau \in A_+$ satisfies $a \in A_+$.

Remark

These conditions are necessary for an ordered algebra with an order unit to have a representation as a function algebra.

Lemma

Let (A, A_+) *be an ordered algebra with algebraic order unit* u*. Put*

$$C := \inf\{r \in \mathbb{R}_+ : \; u^2 \leq ru\}.$$

Then C *is a strictly positive real number satisfying*

$$ua \leq Ca \quad \text{and} \quad au \leq Ca$$

for all $a \in A_+$*.*

Lemma

Let (A, A_+) be an ordered algebra with algebraic order unit u. Put

$$C := \inf\{r \in \mathbb{R}_+ : \ u^2 \le ru\}.$$

Then C is a strictly positive real number satisfying

$$ua \le Ca \quad \textit{and} \quad au \le Ca$$

for all $a \in A_+$.

Example

Let (B, B_+) be as above. Suppose that there exist $u_1, \ldots, u_n$ in B_+ such that every $f \in B$ can be written as

$$f = u_1 f_1 + \cdots + u_n f_n,$$

for some $f_1, \ldots, f_n$ in $C_b(Y)$. Then $u := u_1 + \cdots + u_n$ is an algebraic order unit for B.

Positive functionals and states

Positive functionals and states

Definition

Let (A, A_+) be an ordered algebra with order unit u. A linear functional $\tau \colon A \to \mathbb{C}$ is called

(i) *positive* if $\tau(a) \in \mathbb{R}_+$ for all $a \in A_+$;

(ii) a *state* if τ is positive and $\tau(u) = 1$.

The *state space* of A, denoted by $S(A)$, is the set of all states on A, endowed with the weak topology induced by the elements of A.

Positive functionals and states

Definition

Let (A, A_+) be an ordered algebra with order unit u. A linear functional $\tau\colon A \to \mathbb{C}$ is called

(i) *positive* if $\tau(a) \in \mathbb{R}_+$ for all $a \in A_+$;

(ii) a *state* if τ is positive and $\tau(u) = 1$.

The *state space* of A, denoted by $S(A)$, is the set of all states on A, endowed with the weak topology induced by the elements of A.

The extreme points of the state space of A are called *pure states*, and the set of all pure states on A is denoted by $P(A)$.

Lemma

Let (A, A_+) *be an ordered algebra with algebraic order unit* u*. Then, for every positive functional* τ *on* A*, we have*

$$\frac{\tau(au + ua)^2}{4} \leq \tau(u)\,\tau(a(ua))$$

for all $a \in A$*.*

Lemma

Let (A, A_+) be an ordered algebra with algebraic order unit u. Then, for every positive functional τ on A, we have

$$\frac{\tau(au + ua)^2}{4} \leq \tau(u)\,\tau(a(ua))$$

for all $a \in A$.

Remark

Let τ be a positive functional on an ordered algebra (A, A_+) with an order unit. The Schwartz inequality, meaning that

$$\tau(a)^2 \leq \tau(a^2)$$

for all $a \in A$, need not hold if the order unit is algebraic. In fact, it is possible even for a pure state on an ordered algebra with an algebraic order unit to vanish on the set $A^2 := \{ab : \ a, b \in A\}$.

Essential pure states

Essential pure states

Definition

Let (A, A_+) be an ordered algebra with order unit u. We define the *essential part* of the pure state space of A by

$$P_e(A) := \{\tau \in P(A) : \ \tau(u^2) > 0\},$$

endowed with the topology induced from $S(A)$.

Essential pure states

Definition

Let (A, A_+) be an ordered algebra with order unit u. We define the *essential part* of the pure state space of A by

$$P_e(A) := \{\tau \in P(A) : \ \tau(u^2) > 0\},$$

endowed with the topology induced from $S(A)$.

Proposition

Let (A, A_+) be an ordered algebra with algebraic order unit u. Then $\tau \mapsto \tau(u^2)\,\tau$ is a bijection from $P_e(A)$ onto the set of all non-zero positive multiplicative functionals on A.

Corollary

Let (A, A_+) be an ordered algebra with algebraic order unit u. Then $P_e(A)$ is a σ-compact locally compact Hausdorff space satisfying

$$A_+ = \{a \in A : \ \tau(a) \in \mathbb{R}_+ \textit{ for all } \tau \in P_e(A)\}.$$

Furthermore, for every non-zero $a \in A$, there exists $\tau \in P_e(A)$ such that $\tau(a) \neq 0$. Finally, the closed convex hull of $P_e(A)$ equals $S(A)$.

The weight function

The weight function

Definition

Let (A, A_+) be an ordered algebra with algebraic order unit u.
We define a mapping $\widehat{v}\colon P_e(A) \to \mathbb{R}$ by setting

$$\widehat{v}(\tau) := \frac{1}{\tau(u^2)} \qquad (\tau \in P_e(A))$$

and call it the *weight function* on $P_e(A)$ induced by u.

The weight function

Definition

Let (A, A_+) be an ordered algebra with algebraic order unit u.
We define a mapping $\widehat{v}\colon P_e(A) \to \mathbb{R}$ by setting

$$\widehat{v}(\tau) := \frac{1}{\tau(u^2)} \qquad (\tau \in P_e(A))$$

and call it the *weight function* on $P_e(A)$ induced by u.

Lemma

Let (A, A_+) be an ordered algebra with algebraic order unit u. Then $\widehat{v}$ is a continuous function on $P_e(A)$ satisfying $\widehat{v}(\tau) \geq C^{-1}$ for all $\tau \in P_e(A)$.

Remark

Let (A, A_+) be an ordered algebra with an algebraic order unit. Then the following conditions are equivalent:

(a) $\widehat{v}$ is bounded on $P_e(A)$;

(b) $P_e(A)$ is compact;

(c) A has an approximate identity under $\|\cdot\|_u$;

(d) there exists a constant $C' > 0$ such that $u \leq C'u^2$.

In particular, if the order norm is complete, then A has an approximate identity under $\|\cdot\|_u$ if and only if A has a multiplicative identity.

The order norm

The order norm

Definition

Let (A, A_+) be an ordered algebra with order unit u. The mapping $\|\cdot\|_u \colon A \to \mathbb{R}$, defined by

$$\|a\|_u := \inf\{r \in \mathbb{R}_+ : \ ru - a \in A_+ \text{ and } ru + a \in A_+\},$$

is called the *order norm* on A induced by u.

The order norm

Definition

Let (A, A_+) be an ordered algebra with order unit u. The mapping $\|\cdot\|_u \colon A \to \mathbb{R}$, defined by

$$\|a\|_u := \inf\{r \in \mathbb{R}_+ : \ ru - a \in A_+ \text{ and } ru + a \in A_+\},$$

is called the *order norm* on A induced by u.

Lemma

Let (A, A_+) be an ordered algebra with algebraic order unit u. Then the order norm multiplied by the constant C is an algebra norm on A, and

$$\|a\|_u = \sup_{\tau \in P_e(A)} \widehat{v}(\tau)\, \tau(u^2)\, |\tau(a)|$$

for all $a \in A$.

The representation theorem

The representation theorem

Definition

Let (A, A_+) be an ordered algebra with algebraic order unit u. For each $a \in A$, we define a mapping $\widehat{a}\colon P_e(A) \to \mathbb{C}$ by setting

$$\widehat{a}(\tau) := \tau(u^2)\,\tau(a) \qquad (\tau \in P_e(A))$$

and call it the *u-transform* of a. We write $\widehat{A} := \{\widehat{a} : \ a \in A\}$ and call it the *u-transform algebra* of A.

The representation theorem

Definition

Let (A, A_+) be an ordered algebra with algebraic order unit u. For each $a \in A$, we define a mapping $\widehat{a}\colon P_e(A) \to \mathbb{C}$ by setting

$$\widehat{a}(\tau) := \tau(u^2)\,\tau(a) \qquad (\tau \in P_e(A))$$

and call it the *u-transform* of a. We write $\widehat{A} := \{\widehat{a} : \; a \in A\}$ and call it the *u-transform algebra* of A.

Theorem

Let (A, A_+) be an ordered algebra with algebraic order unit u. Then $a \mapsto \widehat{a}$ is an isometric algebra and order isomorphism from A onto a subalgebra of $C_b^{\widehat{v}}(P_e(A))$ which strongly separates the points of $P_e(A)$.

Proposition

Let (A, A_+) be an ordered algebra with algebraic order unit u. Suppose that the order norm is complete on A. Then $\widehat{A}$ is a closed subalgebra of $C_b^{\widehat{v}}(P_e(A))$ satisfying $C_0^{\widehat{v}}(P_e(A)) \subseteq \widehat{A} \subseteq C_0(P_e(A))$.

Proposition

Let (A, A_+) be an ordered algebra with algebraic order unit u. Suppose that the order norm is complete on A. Then $\widehat{A}$ is a closed subalgebra of $C_b^{\widehat{v}}(P_e(A))$ satisfying $C_0^{\widehat{v}}(P_e(A)) \subseteq \widehat{A} \subseteq C_0(P_e(A))$.

Corollary