The Arveson-Douglas essential normality conjecture

Matthew Kennedy

Carleton University

Aug. 3, 2013

Commuting tuples of operators

Let $A = (A_1, \ldots, A_d)$ be a commuting tuple of Hilbert space operators. In this talk, I hope to convince you that:

Philosophy

To study $A = (A_1, \ldots, A_d)$, we should use ideas from commutative algebra and algebraic geometry.

For monomials in $\mathbb{C}[z] := \mathbb{C}[z_1, \ldots, z_d]$, write

$$z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}, \quad \alpha \in \mathbb{N}^d.$$

For monomials in $\mathbb{C}[z] := \mathbb{C}[z_1, \ldots, z_d]$, write

$$z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}, \quad \alpha \in \mathbb{N}^d.$$

Definition

The **Drury-Arveson Hilbert space** H^2_d is the completion of $\mathbb{C}[z]$ with respect to

$$\langle z^\alpha, z^\beta \rangle = \delta_{\alpha,\beta} \frac{\alpha_1! \cdots \alpha_d!}{(\alpha_1 + \cdots + \alpha_d)!}, \quad \alpha, \beta \in \mathbb{N}^d.$$

For monomials in $\mathbb{C}[z] := \mathbb{C}[z_1, \ldots, z_d]$, write

$$z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}, \quad \alpha \in \mathbb{N}^d.$$

Definition

The **Drury-Arveson Hilbert space** H_d^2 is the completion of $\mathbb{C}[z]$ with respect to

$$\langle z^\alpha, z^\beta \rangle = \delta_{\alpha,\beta} \frac{\alpha_1! \cdots \alpha_d!}{(\alpha_1 + \cdots + \alpha_d)!}, \quad \alpha, \beta \in \mathbb{N}^d.$$

We view $f \in H_d^2$ as an analytic function on the complex unit ball $\mathbb{B}_d$,

$$f(z) = \sum_{\alpha \in \mathbb{N}^d} c_\alpha z^\alpha.$$

Let $M_z = (M_{z_1}, \ldots, M_{z_d})$ denote the d-tuple of coordinate multiplication operators on $\mathbb{C}[z]$,

$$M_{z_i} p(z) = z_i p(z), \quad p \in \mathbb{C}[z].$$

This tuple extends to a contractive d-tuple of operators on the Drury-Arveson Hilbert space H^2_d.

Let $I \lhd \mathbb{C}[z]$ be an ideal. Then I is an invariant subspace for $M_{z_1}, \ldots, M_{z_d}$, so writing $H^2_d = I^\perp \oplus \bar{I}$,

$$M_{z_i} = \begin{pmatrix} A_i & 0 \\ * & * \end{pmatrix}, \quad 1 \leq i \leq d.$$

Let $I \lhd \mathbb{C}[z]$ be an ideal. Then I is an invariant subspace for $M_{z_1}, \dots, M_{z_d}$, so writing $H^2_d = I^\perp \oplus \bar{I}$,

$$M_{z_i} = \begin{pmatrix} A_i & 0 \\ * & * \end{pmatrix}, \quad 1 \le i \le d.$$

Note: We view $I^\perp$ as the completion of $\mathbb{C}[z]/I$.

Let $I \lhd \mathbb{C}[z]$ be an ideal. Then I is an invariant subspace for $M_{z_1}, \ldots, M_{z_d}$, so writing $H^2_d = I^\perp \oplus \bar{I}$,

$$M_{z_i} = \begin{pmatrix} A_i & 0 \\ * & * \end{pmatrix}, \quad 1 \leq i \leq d.$$

Note: We view $I^\perp$ as the completion of $\mathbb{C}[z]/I$.

Observation

The d-tuple $A = (A_1, \ldots, A_d)$ is the (extension of the) d-tuple of coordinate multiplication operators on $\mathbb{C}[z]/I$,

$$A_i \overline{p(z)} = \overline{z_i p(z)}, \quad p \in \mathbb{C}[z].$$

Theorem (Arveson, Müller-Vasilescu)

Every contractive d-tuple of commuting operators $A = (A_1, \dots, A_d)$ *arises in this way.*

Theorem (Arveson, Müller-Vasilescu)

Every contractive d-tuple of commuting operators $A = (A_1, \ldots, A_d)$ *arises in this way.*

We may need to consider vector-valued polynomials. But many interesting problems reduce to the scalar case.

Summary: Let $I \triangleleft \mathbb{C}[z]$ be an ideal. The d-tuple of coordinate multiplication operators $A = (A_1, \ldots, A_d)$ on the quotient $\mathbb{C}[z]/I$ extend to bounded linear operators on $H_I = \overline{\mathbb{C}[z]/I}$.

Summary: Let $I \lhd \mathbb{C}[z]$ be an ideal. The d-tuple of coordinate multiplication operators $A = (A_1, \ldots, A_d)$ on the quotient $\mathbb{C}[z]/I$ extend to bounded linear operators on $H_I = \overline{\mathbb{C}[z]/I}$.

Philosophy

To understand arbitrary commuting tuples of operators, we should try to understand $A = (A_1, \ldots, A_d)$.

The Arveson-Douglas essential normality conjecture

Let $I \triangleleft \mathbb{C}[z]$ be an ideal and let $A = (A_1, \ldots, A_d)$ be the d-tuple of coordinate multiplication operators arising as above on $H_I = \overline{\mathbb{C}[z]/I}$.

Arveson-Douglas Conjecture

We should expect connections between the structure of $A = (A_1, \ldots, A_d)$ and the geometric structure of the variety

$$V(I) = \{\lambda \in \mathbb{C}^d \mid p(\lambda) = 0 \ \forall p \in I\}.$$

.

Let $I \lhd \mathbb{C}[z]$ and $A = (A_1, \ldots, A_d)$ be as above.

Let $I \triangleleft \mathbb{C}[z]$ and $A = (A_1, \ldots, A_d)$ be as above.

Definition

The ideal I is **essentially normal** (resp. p-**essentially normal**) if the self-commutators

$$A_i^* A_j - A_j A_i^*, \quad 1 \leq i, j \leq d$$

are compact (resp. contained in the Schatten p-class).

Let $I \triangleleft \mathbb{C}[z]$ and $A = (A_1, \ldots, A_d)$ be as above.

Definition

The ideal I is **essentially normal** (resp. p-**essentially normal**) if the self-commutators

$$A_i^* A_j - A_j A_i^*, \quad 1 \leq i, j \leq d$$

are compact (resp. contained in the Schatten p-class).

Arveson-Douglas Conjecture

Suppose I is homogeneous (i.e. generated by homogeneous polynomials). Then I is p-essentially normal for every $p > \dim V(I)$.

Consequence: A positive solution to the Arveson-Douglas conjecture would imply the sequence

$$0 \longrightarrow \mathrm{K}(H_I) \longrightarrow \mathrm{C}^*(A_1, \ldots, A_d) + \mathrm{K}\,(H_I) \longrightarrow \mathrm{C}(V(I) \cap \partial\mathbb{B}_d) \longrightarrow 0$$

is exact. The C*-algebra $\mathrm{C}^*(A_1, \ldots, A_d)$ gives rise to an invariant of $V(I)$, conjectured to be the fundamental class of $V(I) \cap \partial\mathbb{B}_d$.

Known Results

Let $I \lhd \mathbb{C}[z]$ be a homogeneous ideal.

Let $I \triangleleft \mathbb{C}[z]$ be a homogeneous ideal.

Theorem (Arveson, 1998)

The conjecture is true for the trivial ideal $I = 0$.

Let $I \lhd \mathbb{C}[z]$ be a homogeneous ideal.

Theorem (Arveson, 1998)

The conjecture is true for the trivial ideal $I = 0$.

Theorem (Arveson, 2003)

The conjecture is true for ideals generated by monomials, i.e. elements of the form z^α for $\alpha \in \mathbb{N}^d$.

The best known results are due to Guo and Wang.

Theorem (Guo-Wang, 2007)

The conjecture is true for ideals generated by a single homogeneous polynomial.

The best known results are due to Guo and Wang.

Theorem (Guo-Wang, 2007)

The conjecture is true for ideals generated by a single homogeneous polynomial.

Theorem (Guo-Wang, 2007)

The conjecture is true for $d \leq 3$.

Theorem (K, 2012)

The conjecture is true for ideals generated by homogeneous polynomials in mutually disjoint variables.

Theorem (K, 2012)

The conjecture is true for ideals generated by homogeneous polynomials in mutually disjoint variables.

For example, the conjecture is true for the ideal

$$I = \langle z_1^2 + z_2^2 - z_3^2,\ z_4^2 + z_5^2 - z_6^2,\ , z_7^2 + z_8^2 - z_9^2 \rangle \lhd \mathbb{C}[z_1, \ldots, z_9].$$

Proof relates the conjecture to the Hilbert space geometry of ideals.

Theorem (K, 2012)

Let $I \triangleleft \mathbb{C}[z]$ be an ideal. The conjecture is true for I if it can be decomposed as

$$I = I_1 + \cdots + I_n,$$

where $I_1, \ldots, I_n$ are ideals satisfying the conjecture with positive angles (in the sense of Friedrichs) between them.

Proof relates the conjecture to the Hilbert space geometry of ideals.

Theorem (K, 2012)

Let $I \triangleleft \mathbb{C}[z]$ be an ideal. The conjecture is true for I if it can be decomposed as

$$I = I_1 + \cdots + I_n,$$

where $I_1, \ldots, I_n$ are ideals satisfying the conjecture with positive angles (in the sense of Friedrichs) between them.

Q: When can we obtain such a decomposition?

Proof relates the conjecture to the Hilbert space geometry of ideals.

Theorem (K, 2012)

Let $I \triangleleft \mathbb{C}[z]$ be an ideal. The conjecture is true for I if it can be decomposed as

$$I = I_1 + \cdots + I_n,$$

where $I_1, \ldots, I_n$ are ideals satisfying the conjecture with positive angles (in the sense of Friedrichs) between them.

Q: When can we obtain such a decomposition?
A (so far): If I is generated by monomials, if I is generated by polynomials in mutually disjoint variables, or if $d \leq 2$ (using Gröbner basis techniques due to Orr Shalit). Plus some other cases.

More recently, we consider the geometry of the variety $V = V(I)$.

More recently, we consider the geometry of the variety $V = V(I)$.

Theorem (K-Shalit, 2012)

Let V and W be homogeneous varities in $\mathbb{C}^d$ with isomorphic coordinate rings. The conjecture is true for V if and only if it is true for W.

More recently, we consider the geometry of the variety $V = V(I)$.

Theorem (K-Shalit, 2012)

Let V and W be homogeneous varities in $\mathbb{C}^d$ with isomorphic coordinate rings. The conjecture is true for V if and only if it is true for W.

Theorem (K-Shalit, 2012)

The conjecture is true for homogeneous varieties in $\mathbb{C}^d$ that are unions of finitely many subspaces.

More recently, we consider the geometry of the variety $V = V(I)$.

Theorem (K-Shalit, 2012)

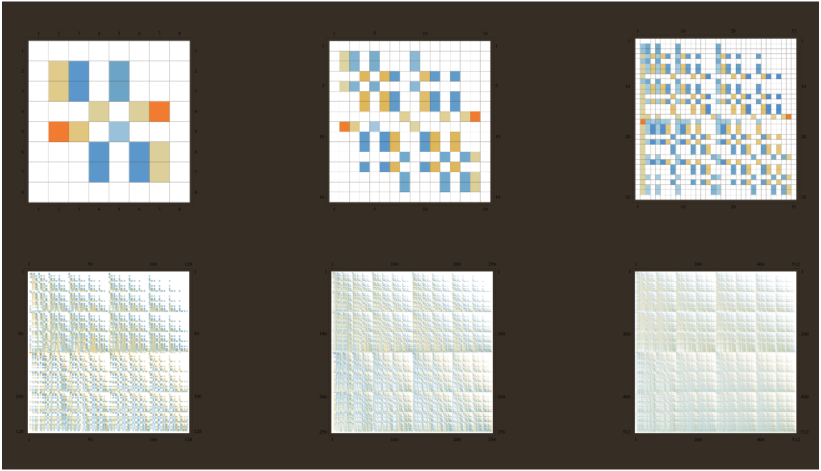
Let V and W be homogeneous varities in $\mathbb{C}^d$ with isomorphic coordinate rings. The conjecture is true for V if and only if it is true for W.

Theorem (K-Shalit, 2012)

The conjecture is true for homogeneous varieties in $\mathbb{C}^d$ that are unions of finitely many subspaces.

We say the conjecture is true for a variety V if it is true for the corresponding ideal $I = I(V)$.

The Arveson-Douglas conjecture is basically a quantitative statement about finite-dimensional matrices. Experiments suggest it is true.



Definition (Arveson, 2011)

A d-tuple of operators $A = (A_1, \ldots, A_d)$ is **hyperrigid** if, the restriction of every unital *-representation π of $\mathrm{C}^*(A_1, \ldots, A_d)$ to

$$\mathrm{span}\{I, A_1, \ldots, A_d\}$$

has a unique unital completely positive extension (namely, π).

Definition (Arveson, 2011)

A d-tuple of operators $A = (A_1, \ldots, A_d)$ is **hyperrigid** if, the restriction of every unital *-representation π of $\mathrm{C}^*(A_1, \ldots, A_d)$ to

$$\mathrm{span}\{I, A_1, \ldots, A_d\}$$

has a unique unital completely positive extension (namely, π).

In other words, if $\Phi = \pi$ on $\mathrm{span}\{I, A_1, \ldots, A_d\}$, then $\Phi = \pi$ on $\mathrm{C}^*(A_1, \ldots, A_d)$.

More recently, we proved the following qualitative reformulation of the conjecture (for $p = \infty$).

More recently, we proved the following qualitative reformulation of the conjecture (for $p = \infty$).

Theorem (Davidson-K-Shalit, 2013)

Let $I \triangleleft \mathbb{C}[z]$ and $A = (A_1, \ldots, A_d)$ be as above. The conjecture is true for I if and only if $\{A_1, \ldots, A_d\}$ is hyperrigid.

Thanks!