Hyper-Invariant subspaces for some compact perturbation of a diagonal operator

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Let $H$ be a complex and separable Hilbert space. We denote the set of bounded operators in $H$ by $\mathcal{B}(H)$. Let $T \in \mathcal{B}(H)$. We denote the commutator of $T$ by

$$\{ T \}' = \{ S \in \mathcal{B}(H), ST = TS \}.$$ 

We denote the spectrum (respectively the point spectrum) of $T$ by $\sigma(T)$ (respectively $\sigma_p(T)$).
**Invariant Subspace Problem**

Let $T \in \mathcal{B}(H)$. Does a non-trivial closed subspace $M \subset H$ (i.e. $M \neq \{0\}$ and $M \neq H$) exist such that $T(M) \subset M$?

**Hyper-Invariant Subspace Problem**

Let $T \in \mathcal{B}(H)$ such that $T \neq \lambda I$. Does a non-trivial closed subspace $M \subset H$ (i.e. $M \neq \{0\}$ and $M \neq H$) exist such that for all $S \in \{T\}'$, we have $S(M) \subset M$?
If $T \neq \lambda I$ and $\sigma_p(T) \neq \emptyset$, then for every $\lambda \in \sigma_p(T)$, Ker($T - \lambda$) is a non trivial hyper invariant subspace. Indeed, let $\lambda \in \sigma_p(T)$, let $x \in \text{Ker}(T - \lambda)$ and $S \in \{T\}'$. Then

$$T(Sx) = S(Tx) = S(\lambda x) = \lambda Sx.$$ 

So $S(\text{Ker}(T - \lambda)) \subset \text{Ker}(T - \lambda)$.

- When $N$ is a normal operator, the spectral Theorem implies the existence of a non trivial hyper invariant subspace for $N$.

- When $K$ is a compact operator, Lomonosov’s Theorem implies the existence of a non trivial hyper invariant subspace for $K$.

- When $N$ is a normal operator and $K$ is a compact operator, we don’t know in general if $N + K$ has a non trivial hyper invariant subspace.
Definition

Let $u, v \in H$. We denote by $u \otimes v$ the rank one operator such that for every $h \in H$ we have

$$u \otimes v(h) = \langle h, v \rangle u.$$ 

Definition

An operator $D \in B(H)$ is diagonal, if there exists an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of $H$, and a bounded sequence of complex number $(\lambda_n)_{n \in \mathbb{N}}$ such that

$$D = \sum_{n \in \mathbb{N}} \lambda_n e_n \otimes e_n$$
If $D$ is a diagonal operator and $u, v \in H$, in general we don’t know if $D + u \otimes v$ has a non trivial hyper invariant subspace.

In 1984, Stampfli constructed a diagonal operator $D$ and two vectors $u, v \in H$ such that $\sigma_p(D + u \otimes v) = \emptyset$.

In 2001, Ionascu gave necessary and sufficient conditions on $D, u, v, \lambda$ in order that $\lambda \in \sigma_p(D + u \otimes v)$.
Let $D = \sum_{n \in \mathbb{N}} \lambda_n e_n \otimes e_n$ be a diagonal operator.
Let $u, v \in H$ such that $D + u \otimes v \neq \lambda I$ and

$$\sum_{n \in \mathbb{N}} |\langle u, e_n \rangle|^3 < \infty \quad \text{and} \quad \sum_{n \in \mathbb{N}} |\langle v, e_n \rangle|^3 < \infty.$$ 

Then $D + u \otimes v$ has a non trivial hyper invariant subspace.
Let $D = \sum_{n \in \mathbb{N}} \lambda_n e_n \otimes e_n$ be a diagonal operator.
Let $u_1, \ldots, u_r, v_1, \ldots, v_r \in H$, such that $D + u_1 \otimes v_1 + \cdots + u_r \otimes v_r \neq \lambda I$ and
\[
\sum_{i=1}^{r} \sum_{n \in \mathbb{N}} |\langle u_i, e_n \rangle| < \infty \quad \text{and} \quad \sum_{i=1}^{r} \sum_{n \in \mathbb{N}} |\langle v_i, e_n \rangle| < \infty.
\]
Then $D + u_1 \otimes v_1 + \cdots + u_r \otimes v_r$ has a non trivial hyper invariant subspace.
Singular Value Decomposition

Let $K \in \mathcal{K}(H)$ be a compact operator. Then there exists a sequence $(s_n)_{n \in \mathbb{N}}$ of positive real numbers such that $\lim_{n \to \infty} s_n = 0$, and there exist two orthonormal families $(u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}}$ of vector in $H$ such that

$$K = \sum_{n \in \mathbb{N}} s_n u_n \otimes v_n.$$
Theorem 1 K. (2013)

Let $D = \sum_{n \in \mathbb{N}} \lambda_n e_n \otimes e_n$ be a diagonal operator.
Let $K = \sum_{n \in \mathbb{N}} s_n u_n \otimes v_n$ be a compact operator.

Suppose that there exists a closed path $\Gamma$ in $\mathbb{C}$ such that

1- There are infinitely many $\lambda_i$ inside and outside $\Gamma$, and
\[ \sigma_p(D) \cap \Gamma = \emptyset, \]

2- The application $A$ is well defined and continuous
\[ A : \Gamma \to \mathcal{K}(H) \]
\[ z \mapsto \sum_{n \in \mathbb{N}} s_n \left( (D - z)^{-1} u_n \right) \otimes \left( (D^* - \overline{z})^{-1} v_n \right). \]

Then $T = D + K$ has a non trivial hyper invariant subspace.
Corollary 2 K. (2013)

Let $D = \sum_{n \in \mathbb{N}} \lambda_n e_n \otimes e_n$ be a diagonal operator.
Let $K = \sum_{n \in \mathbb{N}} s_n u_n \otimes v_n$ be a compact operator.
Let $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ be two sequences of positive real numbers such that for all $n \in \mathbb{N}$, $s_n = a_n b_n$.
Suppose that $D + K \neq \lambda I$ and

$$\sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} |a_n \langle u_n, e_k \rangle| < \infty \quad \text{and} \quad \sum_{j \in \mathbb{N}} \sum_{n \in \mathbb{N}} |b_n \langle e_j, v_n \rangle| < \infty.$$ 

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Fact A
Let $P$ be an orthogonal projection such that
$$\dim(P(H)) = \dim((I - P)(H)) = \infty.$$  
Let $L \in \mathcal{K}(H)$ be a compact operator.  
Then $P + L$ has a non trivial hyper invariant subspace.
This is a well-known fact from the theory of Bochner integral.

**Fact B**

Let $(X, \mathcal{M}, \mu)$ be a measured space.
Let $H$ be a separable Hilbert space.
Let $\mathcal{K}(H)$ be the collection of compact operators on $H$.
Suppose that $F : X \to \mathcal{K}(H)$ is a weakly $\mathcal{M}$-measurable map and

$$
\int_X \| F(x) \| d\mu(x) < \infty.
$$

Then

$$
L = \int_X F(x) d\mu(x)
$$

is a compact operator.
Idea of the proof of Theorem 1

Suppose that $\sigma_p(T) = \emptyset$.

▸ Step 1. Let $W = \cap_{z \in \Gamma} \text{Ran}(D - z)$. For all $z \in \Gamma$ we define

$$B(z) = (I + A(z)(D - z))^{-1}A(z)$$
$$R(z) = (D - z)^{-1} - B(z).$$

For all $w \in W$ we have that

$$(T - z)R(z)w = w.$$  

▸ Step 2. There exists a compact operator $L$ and an orthogonal projection $P$ satisfying the hypothesis of Fact 1, such that for every $w \in W$

$$\frac{1}{2i\pi} \int_{\Gamma} R(z)wdz = \frac{1}{2i\pi} \int_{\Gamma} (D - z)^{-1}wdz - \frac{1}{2i\pi} \int_{\Gamma} B(z)wdz$$

$$= Pw + Lw.$$  

▸ Step 3. $\{T\}' \subset \{P + L\}'$. 

Idea of the proof of Corollary 2

Corollary 2 K. (2013)

Let $D = \sum_{n \in \mathbb{N}} \lambda_n e_n \otimes e_n$ be a diagonal operator.
Let $K = \sum_{n \in \mathbb{N}} s_n u_n \otimes v_n$ be a compact operator.
Let $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ be two sequences of positive real numbers such that for all $n \in \mathbb{N}$, $s_n = a_n b_n$.
Suppose that $D + K \neq \lambda I$ and

$$\sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} |a_n \langle u_n, e_k \rangle| < \infty \quad \text{and} \quad \sum_{j \in \mathbb{N}} \sum_{n \in \mathbb{N}} |b_n \langle e_j, v_n \rangle| < \infty.$$ 

Then $T = D + K$ has a non trivial hyper invariant subspace.
Lemma 3 K. (2013)

Let \((a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}\) be two sequences of positive real numbers. Let \((e_n)_{n \in \mathbb{N}}\) be an orthonormal basis of \(H\). Let \((u_n)_{n \in \mathbb{N}}, (v_n)_{n \in \mathbb{N}}\) be two family of orthonormal vector in \(H\). Let \((\lambda_n)_{n \in \mathbb{N}}\) be a bounded sequence of complex numbers.

Suppose that

\[
\sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} |a_n \langle u_n, e_k \rangle| < \infty \quad \text{and} \quad \sum_{j \in \mathbb{N}} \sum_{n \in \mathbb{N}} |b_n \langle e_j, v_n \rangle| < \infty.
\]

Then for almost every \(x \in \mathbb{R}\), we have that

\[
g(x) = \sum_{k \in \mathbb{N}} \sum_{n \in \mathbb{N}} \frac{|a_n \langle u_n, e_k \rangle|^2}{\text{Re}(\lambda_k - x)^2} < \infty
\]

and

\[
h(x) = \sum_{j \in \mathbb{N}} \sum_{n \in \mathbb{N}} \frac{|b_n \langle e_j, v_n \rangle|^2}{\text{Re}(\lambda_k - x)^2} < \infty.
\]
Denote by $z = x + iy$. For all $z \in \mathbb{C}$ we have that $\|A(z)\| \leq g(x)h(x)$. 
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Suppose that \( D + K \neq \lambda I \) and

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Then \( T = D + K \) has a non trivial hyper invariant subspace.