Hyper-Invariant subspaces for some compact perturbation of a diagonal operator

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Let H be a complex and separable Hilbert space.
We denote the set of bounded operators in H by $\mathcal{B}(H)$.
Let $T \in \mathcal{B}(H)$. We denote the commutator of T by

$$\{T\}' = \{S \in \mathcal{B}(H), ST = TS\}.$$

We denote the spectrum (respectively the point spectrum) of T by $\sigma(T)$ (respectively $\sigma_p(T)$).

Invariant Subspace Problem

Let $T \in \mathcal{B}(H)$. Does a non trivial closed subspace $M \subset H$ (i.e. $M \neq \{0\}$ and $M \neq H$) exist such that $T(M) \subset M$?

Hyper-Invariant Subspace Problem

Let $T \in \mathcal{B}(H)$ such that $T \neq \lambda I$. Does a non trivial closed subspace $M \subset H$ (i.e. $M \neq \{0\}$ and $M \neq H$) exist such that for all $S \in \{T\}'$, we have $S(M) \subset M$?

- If $T \neq \lambda I$ and $\sigma_p(T) \neq \emptyset$, then for every $\lambda \in \sigma_p(T)$, $\mathrm{Ker}(T-\lambda)$ is a non trivial hyper invariant subspace. Indeed, let $\lambda \in \sigma_p(T)$, let $x \in \mathrm{Ker}(T-\lambda)$ and $S \in \{T\}'$. Then

$$T(Sx) = S(Tx) = S(\lambda x) = \lambda Sx.$$

  So $S(\mathrm{Ker}(T-\lambda)) \subset \mathrm{Ker}(T-\lambda)$.
- When N is a normal operator, the spectral Theorem implies the existence of a non trivial hyper invariant subspace for N.
- When K is a compact operator, Lomonosov's Theorem implies the existence of a non trivial hyper invariant subspace for K.
- When N is a normal operator and K is a compact operator, we don't know in general if $N+K$ has a non trivial hyper invariant subspace.

Definition

Let $u, v \in H$. We denote by $u \otimes v$ the rank one operator such that for every $h \in H$ we have

$$u \otimes v(h) = \langle h, v \rangle \, u.$$

Definition

An operator $D \in \mathcal{B}(H)$ is *diagonal*, if there exists an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H, and a bounded sequence of complex number $(\lambda_n)_{n\in\mathbb{N}}$ such that

$$D = \sum_{n\in\mathbb{N}} \lambda_n e_n \otimes e_n$$

- If D is a diagonal operator and $u, v \in H$, in general we don't know if $D + u \otimes v$ has a non trivial hyper invariant subspace.
- In 1984, Stampfli constructed a diagonal operator D and two vectors $u, v \in H$ such that $\sigma_p(D + u \otimes v) = \emptyset$.
- In 2001, Ionascu gave necessary and sufficient conditions on D, u, v, λ in order that $\lambda \in \sigma_p(D + u \otimes v)$.

Foias, Jung, Ko, Pearcy (2007)

Let $D = \sum_{n\in\mathbb{N}} \lambda_n e_n \otimes e_n$ be a diagonal operator.
Let $u, v \in H$ such that $D + u \otimes v \neq \lambda I$ and

$$\sum_{n\in\mathbb{N}} |\langle u, e_n\rangle|^{\frac{2}{3}} < \infty \quad \text{and} \quad \sum_{n\in\mathbb{N}} |\langle v, e_n\rangle|^{\frac{2}{3}} < \infty.$$

Then $D + u \otimes v$ has a non trivial hyper invariant subspace.

Fang, Xia (2012)

Let $D = \sum_{n\in\mathbb{N}} \lambda_n e_n \otimes e_n$ be a diagonal operator.
Let $u_1 \ldots, u_r, v_1, \ldots, v_r \in H$,
such that $D + u_1 \otimes v_1 + \cdots + u_r \otimes v_r \neq \lambda I$ and

$$\sum_{i=1}^{r} \sum_{n\in\mathbb{N}} |\langle u_i, e_n\rangle| < \infty \quad \text{and} \quad \sum_{i=1}^{r} \sum_{n\in\mathbb{N}} |\langle v_i, e_n\rangle| < \infty.$$

Then $D + u_1 \otimes v_1 + \cdots + u_r \otimes v_r$ has a non trivial hyper invariant subspace.

Singular Value Decomposition

Let $K \in \mathcal{K}(H)$ be a compact operator. Then there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of positive real numbers such that $\lim_{n\to\infty} s_n = 0$, and there exist two orthonormal families $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}$ of vector in H such that

$$K = \sum_{n\in\mathbb{N}} s_n u_n \otimes v_n.$$

Theorem 1 K. (2013)

Let $D = \sum_{n\in\mathbb{N}} \lambda_n e_n \otimes e_n$ be a diagonal operator.
Let $K = \sum_{n\in\mathbb{N}} s_n u_n \otimes v_n$ be a compact operator.
Suppose that there exists a closed path Γ in $\mathbb{C}$ such that

1- There are infinitely many λ_i inside and outside Γ, and $\sigma_p(D) \cap \Gamma = \emptyset$,

2- The application A is well defined and continuous

$$\begin{aligned} A : \Gamma &\to \mathcal{K}(H) \\ z &\mapsto \sum_{n\in\mathbb{N}} s_n \left((D-z)^{-1} u_n\right) \otimes \left((D^* - \overline{z})^{-1} v_n\right). \end{aligned}$$

Then $T = D + K$ has a non trivial hyper invariant subspace.

Corollary 2 K. (2013)

Let $D = \sum_{n\in\mathbb{N}} \lambda_n e_n \otimes e_n$ be a diagonal operator.
Let $K = \sum_{n\in\mathbb{N}} s_n u_n \otimes v_n$ be a compact operator.
Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ be two sequences of positive real numbers such that for all $n \in \mathbb{N}$, $s_n = a_n b_n$.
Suppose that $D + K \neq \lambda I$ and

$$\sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}} |a_n \langle u_n, e_k \rangle | < \infty \quad \text{and} \quad \sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{N}} |b_n \langle e_j, v_n \rangle | < \infty.$$

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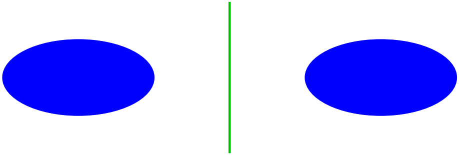

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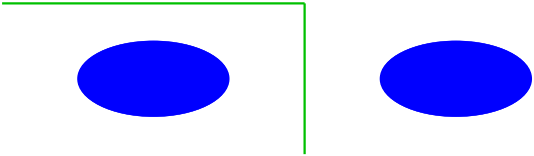

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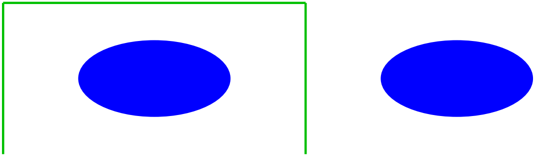

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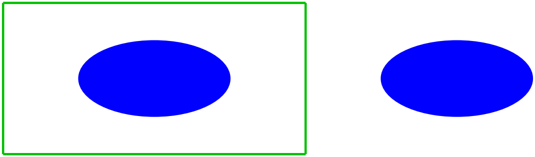

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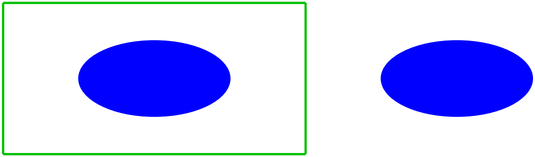

Preliminary results

Fact A

Let P be an orthogonal projection such that
$\dim\left(P(H)\right) = \dim\left((I - P)(H)\right) = \infty$.
Let $L \in \mathcal{K}(H)$ be a compact operator.
Then $P + L$ has a non trivial hyper invariant subspace.

This is a well-known fact from the theory of Bochner integral.

Fact B

Let $(X, \mathcal{M}, \mu)$ be a measured space.
Let H be a separable Hilbert space.
Let $\mathcal{K}(H)$ be the collection of compact operators on H.
Suppose that $F : X \to \mathcal{K}(H)$ is a weakly $\mathcal{M}$-measurable map and

$$\int_X \|F(x)\| \mathrm{d}\mu(x) < \infty.$$

Then

$$L = \int_X F(x) \mathrm{d}\mu(x)$$

is a compact operator.

Idea of the proof of Theorem 1

Suppose that $\sigma_p(T) = \emptyset$.

- Step 1. Let $W = \cap_{z \in \Gamma} \mathrm{Ran}(D - z)$. For all $z \in \Gamma$ we define

$$B(z) = (I + A(z)(D - z))^{-1} A(z)$$
$$R(z) = (D - z)^{-1} - B(z).$$

  For all $w \in W$ we have that

$$(T - z) R(z) w = w.$$

- Step 2. There exists a compact operator L and an orthogonal projection P satisfying the hypothesis of Fact 1, such that for every $w \in W$

$$\frac{1}{2i\pi} \int_\Gamma R(z) w \mathrm{d}z = \frac{1}{2i\pi} \int_\Gamma (D - z)^{-1} w \mathrm{d}z - \frac{1}{2i\pi} \int_\Gamma B(z) w \mathrm{d}z$$
$$= Pw + Lw.$$

- Step 3. $\{T\}' \subset \{P + L\}'$.

Idea of the proof of Corollary 2

Corollary 2 K. (2013)

Let $D = \sum_{n\in\mathbb{N}} \lambda_n e_n \otimes e_n$ be a diagonal operator.
Let $K = \sum_{n\in\mathbb{N}} s_n u_n \otimes v_n$ be a compact operator.
Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ be two sequences of positive real numbers such that for all $n \in \mathbb{N}$, $s_n = a_n b_n$.
Suppose that $D + K \neq \lambda I$ and

$$\sum_{k\in\mathbb{N}} \sum_{n\in\mathbb{N}} |a_n \langle u_n, e_k \rangle| < \infty \quad \text{and} \quad \sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{N}} |b_n \langle e_j, v_n \rangle| < \infty.$$

Then $T = D + K$ has a non trivial hyper invariant subspace.

Lemma 3 K. (2013)

Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be two sequences of positive real numbers.
Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H.
Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ be two family of orthonormal vector in H.
Let $(\lambda_n)_{n\in\mathbb{N}}$ be a bounded sequence of complex numbers.
Suppose that

$$\sum_{k\in\mathbb{N}}\sum_{n\in\mathbb{N}} |a_n \langle u_n, e_k\rangle| < \infty \quad \text{and} \quad \sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{N}} |b_n \langle e_j, v_n\rangle| < \infty.$$

Then for almost every $x \in \mathbb{R}$, we have that

$$g(x) = \sum_{k\in\mathbb{N}}\sum_{n\in\mathbb{N}} \frac{|a_n \langle u_n, e_k\rangle|^2}{\mathrm{Re}(\lambda_k - x)^2} < \infty$$

and

$$h(x) = \sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{N}} \frac{|b_n \langle e_j, v_n\rangle|^2}{\mathrm{Re}(\lambda_k - x)^2} < \infty.$$

Denote by $z = x + \mathrm{i}y$. For all $z \in \mathbb{C}$ we have that $\|A(z)\| \leq g(x)h(x)$.

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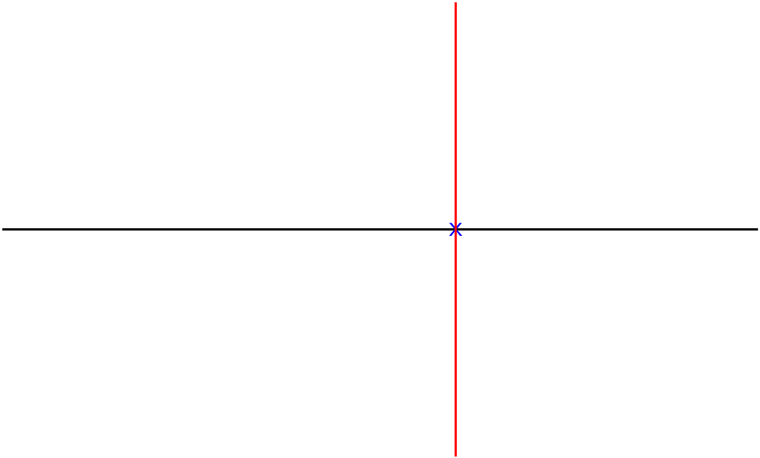

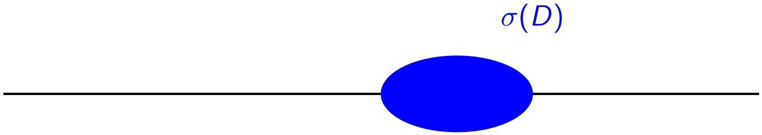
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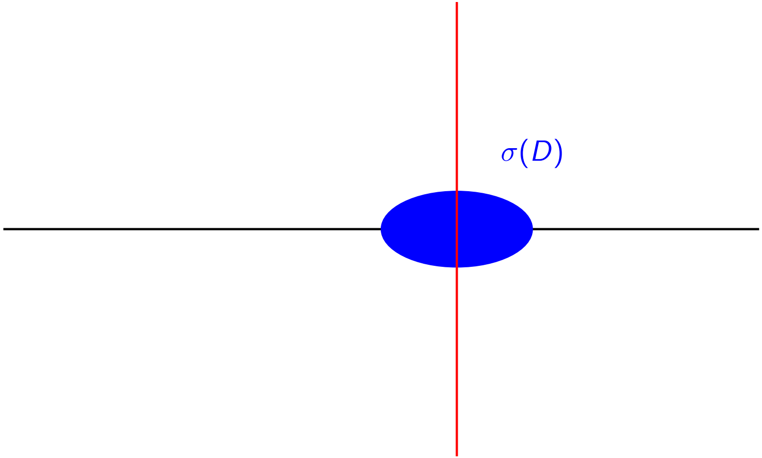
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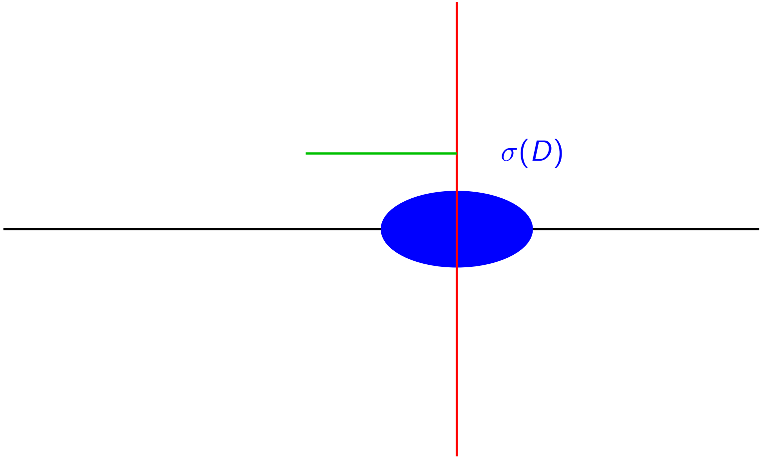
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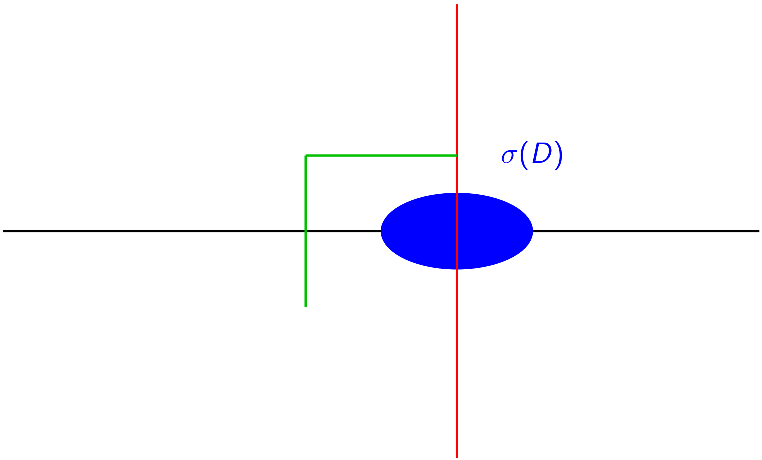
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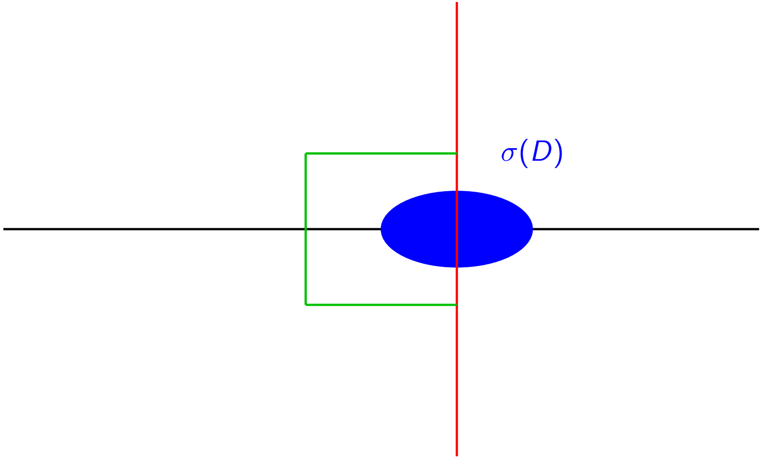
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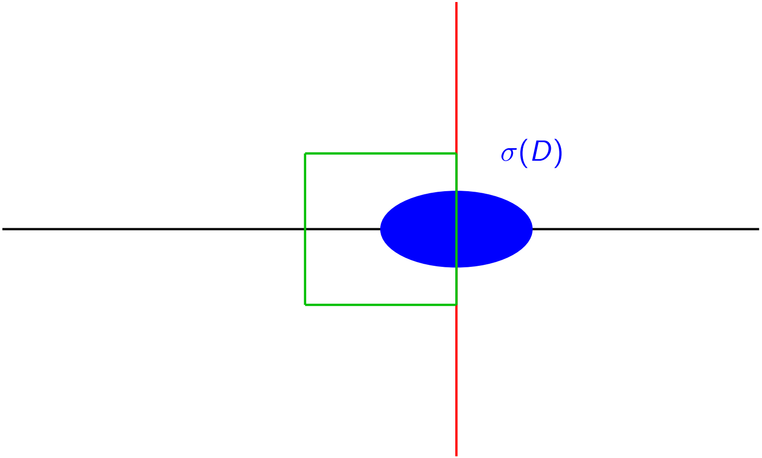
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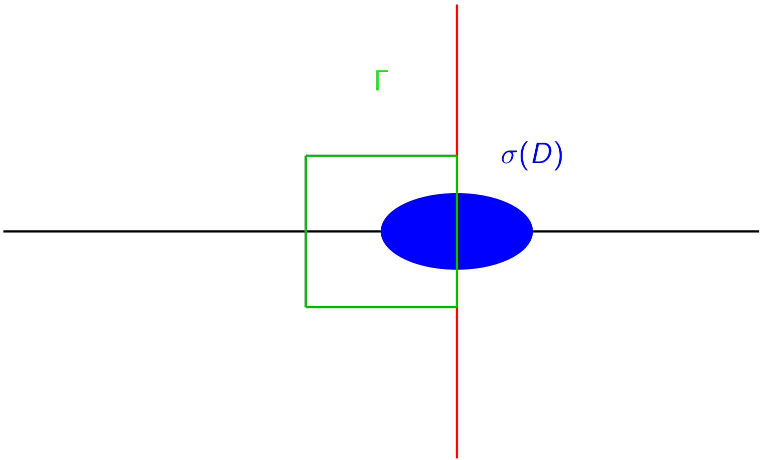
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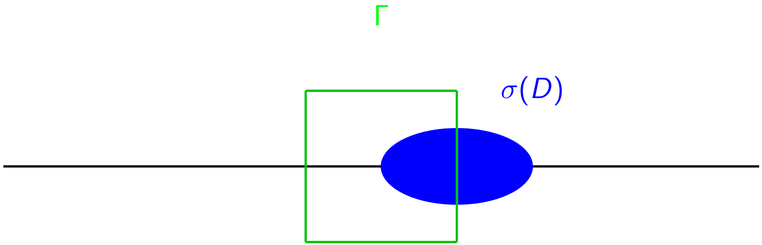
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