Dual algebras and A-measures.

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Joint work with Krzysztof Rudol

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- the application of dual algebras in functional calculus for bounded operators in Hilbert spaces
- connections with the Corona problem

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- a general positive solution for A - measures problem
- the duality of $H^\infty(G)$ algebra for some classes of bounded domains $G \subset \mathbb{C}^n$

Definition

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We assume $\sigma(A) = X$.

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- for $G \subset X$ we denote by $\mathcal{M}_G$ the band generated by G i.e. the smallest band containing all measures representing for points in G
- if G is a Gleason part then $\mathcal{M}_G$ is a reducing band

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- for $\mu \in M(X)$ there is a unique measure $\tilde{\mu} \in M(Y) = C(Y)^*$ such that $\langle F, \mu \rangle = \int F \, d\tilde{\mu}$ for all $F \in C(Y)$

Theorem

If G is a Gleason part of A then the weak-star closure $\overline{G}^{ws}$ of G in Y is a closed-open subset of Y. Moreover

$$Y \setminus \overline{G}^{ws} = \overline{X \setminus G}^{ws}, \quad (\overline{\mathcal{M}_G}^{ws})^s = \overline{(\mathcal{M}_G^s)}^{ws}, \quad \overline{\mathcal{M}_G}^{ws} = M(\overline{G}^{ws}),$$

and $\overline{\mathcal{M}_G}^{ws}$ is a reducing band for A^{**}.

Corollary

There exists a characteristic function $F_0 \in A^{**}$ vanishing exactly on $Y \setminus \overline{G}^{ws}$ and the projection associated with the decomposition $M(Y) = \overline{\mathcal{M}_G}^{ws} + \overline{\mathcal{M}_G^s}^{ws}$ is exactly the multiplication by F_0.

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Corollary

If G is a Gleason part of a function algebra A, $x \in G$ and μ_x is any its representing measure, then μ_x is concentrated on the weak-star closure of G.

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Proposition

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Corollary

G is a subset of the spectrum of $H^\infty(\mathcal{M}_G)$

Theorem

If G is a Gleason part of A, then $H^\infty(\mathcal{M}_G)$ satisfies the domination condition:

$$\|f\| = \sup_{x \in G} |f(x)| \qquad \text{for any} \quad f \in H^\infty(\mathcal{M}_G)$$

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- By the weak-star density of A in $H^\infty(\mathcal{M}_G)$, the value $f(z)$ does not depend on the choice of representing measure.
- So the elements of $H^\infty(\mathcal{M}_G)$ can be regarded as functions on G.

Proposition

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Open problem

Is $\sigma(A^{**}) = Y/_{(A^{**})^\perp}$, where Y is the spectrum of $C(X)^{**}$?

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Is $\sigma(A^{**}) = Y/_{(A^{**})^\perp}$, where Y is the spectrum of $C(X)^{**}$?

Consequences

If the above open problem would have a positive solution, then the Corona problem would have a positive solution for the case when $H^\infty(G)$ and $H^\infty(\mathcal{M}_G)$ are isometrically isomorphic.

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- We say that a measure $\mu \in M(X)$ is an *A-measure* (or analytic measure, or a Henkin measure) with respect to the the set Q if $\int u_n \, d\mu \to 0$ whenever $\{u_n\}_{n=1}^{\infty} \subset A$ is a bounded sequence converging to 0 pointwise on Q.

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A-measures problem for the algebra A at the points of Q

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Another formulation

Is any measure which is absolutely continuous with respect to a positive A-measure, itself an A-measure?

Theorem

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Corollary

The A-measures problem at the points of $Q = G$ for $A(G)$ has a positive solution if G is either a strictly pseudoconvex set in $\mathbb{C}^n$, or a Carthesian product of a finite number of such domains.

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Theorem

A-measures problem for the algebra $A = H^\infty(G)$ at all points of a countable union Q of its arbitrary Gleason parts has positive solution. In particular, if G is a star-shaped domain in $\mathbb{C}^n$ such that $\overline{G}$ is the spectrum of $A(G)$, then A-measures problem for $H^\infty(G)$ at all points of G has positive solution.

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Both above cases are covered by our results.

THANK YOU!