Dual algebras and A-measures.

Marek Kosiek

Joint work with Krzysztof Rudol
$A$ - an arbitrary function algebra
The main subject of our investigation:
Properties of the spectrum of $A^{**}$. 

Motivation:
- $A$ - measures problem
- $G \subset \mathbb{C}^n$
- The algebra $H^\infty(G)$ is a dual algebra
- The application of dual algebras in functional calculus for bounded operators in Hilbert spaces
- Connections with the Corona problem

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- a general positive solution for A-measures problem
- the duality of $H^\infty(G)$ algebra for some classes of bounded domains $G \subset \mathbb{C}^n$
Definition

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A contains constants and separates the points of $X$.
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\phi \sim \psi \quad \overset{\text{df}}{\iff} \quad \| \phi - \psi \| < 2
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We assume $\sigma(A) = X$.  

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every measure \( \mu \in \mathcal{M}(X) \) has a unique Lebesque decomposition \( \mu = \mu_M + \mu_S \) where \( \mu_M \in \mathcal{M} \) and \( \mu_S \) is singular to each measure in \( \mathcal{M} \)
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\( \mathcal{M} \) is a reducing band (with respect to \( A \)) if \( \mu \in A^\perp \implies \mu_M \in A^\perp \)
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\( \nu \) is a representing measure for \( x \in X = \sigma(A) \) if \( f(x) = \int f \, d\nu \) for \( f \in A \).
• $\mathcal{M} \subset M(X) = C(X)^*$ is a band if it is a closed subspace and $\mu \in \mathcal{M}, \nu \ll |\mu| \implies \nu \in \mathcal{M}$

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• for $G \subset X$ we denote by $\mathcal{M}_G$ the band generated by $G$ i.e. the smallest band containing all measures representing for points in $G$
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if \( G \) is a Gleason part then \( \mathcal{M}_G \) is a reducing band
\((C(X))^* \approx M(X) \approx C(Y)\) for some hyperstonean compact space \(Y\)
(\(C(X)^*\))^* = M(X)^* \cong C(Y)\) for some hyperstonean compact space \(Y\)

each \(f \in C(X)\) can be treated as a functional on \(M(X)\) and consequently as an element of \(C(Y)\) by the formula

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for \(\mu \in M(X)\) there is a unique measure \(\tilde{\mu} \in M(Y) = C(Y)^*\) such that \(\langle F, \mu \rangle = \int F \, d\tilde{\mu}\) for all \(F \in C(Y)\)
Theorem

If \( G \) is a Gleason part of \( A \) then the weak-star closure \( \overline{G}^{ws} \) of \( G \) in \( Y \) is a closed-open subset of \( Y \). Moreover

\[
Y \setminus \overline{G}^{ws} = X \setminus \overline{G}^{ws}, \quad (\overline{\mathcal{M}_G}^{ws})^s = (\overline{\mathcal{M}_G^s})^{ws}, \quad \overline{\mathcal{M}_G}^{ws} = \mathcal{M}(\overline{G}^{ws}),
\]

and \( \overline{\mathcal{M}_G}^{ws} \) is a reducing band for \( A^{**} \).
Corollary

There exists a characteristic function $F_0 \in A^{**}$ vanishing exactly on $Y \setminus \overline{G}^{ws}$ and the projection associated with the decomposition $M(Y) = \overline{\mathcal{M}}_G^{ws} + \overline{\mathcal{M}}_S^{ws}$ is exactly the multiplication by $F_0$. 
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Corollary

If $G$ is a Gleason part of a function algebra $A$, $x \in G$ and $\mu_x$ is any its representing measure, then $\mu_x$ is concentrated on the weak-star closure of $G$. 
- $G$ - a Gleason part of $A$
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**Proposition**

$H^\infty(\mathcal{M}_G)$ is isometrically isomorphic to $A^{**}/\mathcal{M}_G^\perp \cap A^{**}$
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**Corollary**

$G$ is a subset of the spectrum of $H^\infty(\mathcal{M}_G)$
Theorem

If $G$ is a Gleason part of $A$, then $H^\infty(\mathcal{M}_G)$ satisfies the domination condition:

$$\|f\| = \sup_{x \in G} |f(x)| \quad \text{for any} \quad f \in H^\infty(\mathcal{M}_G)$$
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The band $\mathcal{M}_G$ is equal to the norm closed linear span of all representing measures for points in $G$, taken in the quotient space $M(X)/A^\perp$. 
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- By the weak-star density of $A$ in $H^\infty(\mathcal{M}_G)$, the value $f(z)$ does not depend on the choice of representing measure.
- So the elements of $H^\infty(\mathcal{M}_G)$ can be regarded as functions on $G$. 

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If $G$ is a bounded domain in $\mathbb{C}^n$ and $f \in H^\infty(M_G)$ then the defined above $z \rightarrow f(z)$ is a bounded analytic function of $z \in G$. 

Proposition

If $G$ is a star-shaped domain in $\mathbb{C}^n$ such that $G$ is the spectrum of $A(G)$, then the algebras $H^\infty(G)$ and $H^\infty(M_G)$ are isometrically isomorphic. Hence $H^\infty(G)$ is a dual algebra.

Open problem

Is $\sigma(A^{**}) = Y/ (A^{**})^\perp$, where $Y$ is the spectrum of $C(X)^{**}$?

Consequences

If the above open problem would have a positive solution, then the Corona problem would have a positive solution for the case when $H^\infty(G)$ and $H^\infty(M_G)$ are isometrically isomorphic.
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\[ Q = \bigcup_{\alpha} G_{\alpha}, \quad \text{where } G_{\alpha} \text{ is a Gleason part of } A \]
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We say that a measure \( \mu \in M(X) \) is an \( A \)-measure (or analytic measure, or a Henkin measure) with respect to the set \( Q \) if \( \int u_n \, d\mu \to 0 \) whenever \( \{u_n\}_{n=1}^\infty \subset A \) is a bounded sequence converging to 0 pointwise on \( Q \).
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**A-measures problem for the algebra \( A \) at the points of \( Q \)**

Does the absolute continuity of a measure \( \mu \) on \( X \) with respect to some representing measure of a point \( x \in Q \) imply that \( \mu \) is an A-measure?
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**Another formulation**

Is any measure which is absolutely continuous with respect to a positive A-measure, itself an A-measure?
Theorem
If $A$ is a function algebra on $X$ and $Q \subset X$ is equal to a countable union of its Gleason parts, then A-measures problem for the algebra $A$ at the points of $Q$ has a positive solution.

Corollary
The A-measures problem at the points of $Q = G$ for $A(G)$ has a positive solution if $G$ is either a strictly pseudoconvex set in $\mathbb{C}^n$, or a Cartesian product of a finite number of such domains. This includes polydiscs, polydomains (products of bounded plane domains), but also products of balls with polydiscs.

Theorem
A-measures problem for the algebra $A = \mathcal{H}\infty(G)$ at all points of a countable union $Q$ of its arbitrary Gleason parts has positive solution. In particular, if $G$ is a star-shaped domain in $\mathbb{C}^n$ such that $G$ is the spectrum of $A(G)$, then A-measures problem for $\mathcal{H}\infty(G)$ at all points of $G$ has positive solution.
Theorem

If $A$ is a function algebra on $X$ and $Q \subset X$ is equal to a countable union of its Gleason parts, then $A$-measures problem for the algebra $A$ at the points of $Q$ has a positive solution.

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Both above cases are covered by our results.
THANK YOU!