Criterion of irreducibility of $C^*$-algebra generated by mapping

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• $C^*$-algebra $C^*_\varphi(X)$ generated by mapping.
• Some properties of $C^*_\varphi(X)$.
• Criterion of irreducibility of the natural representation of $C^*_\varphi(X)$.
• Basic invariant subspaces of $C^*_\varphi(X)$.
• Some examples of $C^*_\varphi(X)$ illustrating the basic invariant subspaces.
Let $X$ be an arbitrary countable set. Mapping $\varphi : X \rightarrow X$ generates oriented graph $(X, \varphi)$ with vertices in the elements of $X$ and edges $(x, \varphi(x))$. We assume the cardinalities of preimages are bounded under the action of mapping $\varphi$. 
Let \( X \) be an arbitrary countable set. Mapping \( \varphi : X \to X \) generates oriented graph \((X, \varphi)\) with vertices in the elements of \( X \) and edges \((x, \varphi(x))\). We assume the cardinalities of preimages are bounded under the action of mapping \( \varphi \).

Mapping \( \varphi \) induces the mapping

\[
T_\varphi : l^2(X) \to l^2(X); \quad T_\varphi f = f \circ \varphi
\]

and hence the family of partial isometries \( \{U_k\} \) participating in decomposition of operator \( T_\varphi \),

\[
T_\varphi = U_1 + \sqrt{2}U_2 + \cdots + \sqrt{m}U_m + \cdots.
\]
Definition

Under $C^*$-algebra generated by mapping $\varphi$ we mean the $C^*$-algebra $C^*_\varphi(X)$ generated by the set of partial isometries $\{U_k\}_{k=1}^\infty$. 

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If the cardinalities of preimages are bounded in common under the action of mapping $\varphi$ ($\sup_{y \in X} \text{card} \varphi^{-1}(y) = m < \infty$), then operator $T_\varphi$ is bounded. In this case $C_\varphi^*(X)$ is a singly generated algebra with the generator $T_\varphi$. 
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The family of partial isometries satisfy the equalities:

$$U_1^*U_1 + U_2^*U_2 + \cdots + U_m^*U_m + \cdots = P;$$

$$U_1U_1^* + U_2U_2^* + \cdots + U_mU_m^* + \cdots = Q;$$

where $P$ and $Q$ are projections defined by the mapping $\varphi$. 

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We call an element from $\{U_k\}_{k=1}^{\infty} \cup \{U_k^*\}_{k=1}^{\infty}$ the primary monomial. We define

$$\text{ind } U_k = -1 \quad \text{and} \quad \text{ind } U_k^* = 1$$

We call $V$ the monomial if it is any finite product of primary monomials not identically zero,

$$V = \prod_{k=1}^{m} U'_{j_k}, \quad U'_{j_k} \in \{U_{j_k} \cup U_{j_k}^*\}, \quad \text{and}$$

$$\text{ind } V = \sum_{k=l}^{m} \text{ind } U'_{j_k}.$$
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V = \prod_{k=1}^m U_j^l, \quad U_j^l \in \{ U_j \cup U_j^* \}, \quad \text{and}
\]

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\text{ind } V = \sum_{k=1}^m \text{ind } U_j^l.
\]

We assume \( X \) and \( \varphi \) to satisfy the following conditions:

- \( \sup_{y \in X} \text{card}\varphi^{-1}(y) = m < \infty \);
- graph \((X, \varphi)\) is connected;
- there is no element in \( X \) such that \( \varphi^n(y) = y \) for some \( n \in \mathbb{N} \).
In a case of absence of cyclic elements the index of monomial is well-defined. Hence we can equip $C^*_\varphi(X)$ with a circle action $\alpha$.

$$C^*_\varphi(X) = \bigoplus_{n \in \mathbb{Z}} C^*_\varphi, n.$$ 

The linear combinations of monomials of index $n$ are dense in $n^{th}$ spectral subspace

$$C^*_\varphi, n = \{ A \in C^*_\varphi(X) : \alpha_z(A) = z^n A \text{ for } z \in S^1 \}.$$
Structure of $C^*_\varphi(X)$

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The fixed point subalgebra $C^*_{\varphi,0}$ (generated by monomials of index zero) is $AF$-algebra.
Mapping $\varphi$ induces the partial order in $X$. We will write
\[ x \prec y \quad \text{if there is an} \quad m \in \mathbb{Z}_+ \quad \text{such that} \quad \varphi^m(y) = x \]
and
\[ x \sim y \quad \text{if there is an} \quad m \in \mathbb{Z}_+ \quad \text{such that} \quad \varphi^m(x) = \varphi^m(y). \]
The Hilbert space $l^2(X)$ has natural basis $\{ e_x \}_{x \in X}$, $e_x(y) = \delta_{x,y}$. 
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The Hilbert space \( l^2(X) \) has natural basis \( \{e_x\}_{x \in X}, \quad e_x(y) = \delta_{x,y} \). We can spread the order on the natural basis by setting
\[
e_x \prec e_y \quad \text{if } x \prec y
\]
and
\[
e_x \sim e_y \quad \text{if } x \sim y.
\]
Theorem
Let \((X, \varphi)\) be a connected graph, \(\{e_x\}\) a natural basis in \(l^2(X)\). Then the following are equivalent:

- \(C^*(X)\) is irreducible on \(l^2(X)\);
- if \((Ve_x, e_x) = (Ve_y, e_y)\) for all monomials \(V \in C^*(X)\), then \(x = y\).
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Let \(\{e_x\}\) be a natural basis in \(l^2(X)\). We will say that the basis elements \(e_x\) and \(e_y\) satisfy the condition \(\omega (e_x \omega e_y)\) if \((V e_x, e_x) = (V e_y, e_y)\) for all \(V \in C^*_\varphi(X)\).
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\(C^*_\varphi(X)\) is irreducible \iff \(e_x \omega e_x\) only for all \(x \in X\).
Let $E(x) = \{x' \in \varphi^{-1}(\varphi(x)) : e_x \varnothing e_{x'}\}$. 
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**Lemma**

Let $e_x \omega e_y$. Then

- for every $x' \in \varphi^{-1}(x)$ such an $y' \in \varphi^{-1}(y)$ exists that $e_{x'} \omega e_{y'}$;
- $\text{card } E(x') = \text{card } E(y')$. 

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Suppose there exist basis elements satisfying the condition $\omega$. What invariant subspaces can $C^*_\varphi(X)$ have?

- $e_x \omega e_y, \quad e_x \sim e_y$;
- $e_x \omega e_y, \quad e_x \asymp e_y$. 
Let $Y$ be a countable set with a mapping $\psi : Y \longrightarrow Y \cup \{\emptyset\}$ such that there exists the minimal element $y_0$ (for all $y \in Y$ there is an $m$ such that $\varphi^m(y) = y_0$) and $\psi(y_0) = \emptyset$. 
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**Proposition**

Let $\psi : Y \to Y \cup \{\emptyset\}$ be a mapping with minimal element generating $C^*$-algebra $\mathcal{A}_\psi$.

Then there exist a set $X$ and mapping $\varphi : X \to X$ such that:

- $l^2(Y) \hookrightarrow l^2(X)$;
- $\text{Im}(l^2(Y))$ is invariant for $C^*_\varphi(X)$;
- $\Omega : C^*_\varphi(X) \to \mathcal{A}_\psi$ — surjective $*$-homomorphism.
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**Proposition**

Let $e_x \sim e_y$ and $e_x \sim e_y$. Then there exist a set $Y$ and mapping $\psi : Y \rightarrow Y \cup \{\emptyset\}$ with minimal element such that:

- $l^2(Y) \hookrightarrow l^2(X)$;
- $\text{Im}(l^2(Y))$ is invariant for $C^*_\varphi(X)$;
- $\Omega : C^*_\varphi(X) \rightarrow \mathfrak{A}_\psi$ — surjective $*$-homomorphism.
Theorem

Let \( \varphi \) be a mapping such that all \( e_x \omega e_y \) are equivalent. Then

\[
l^2(X) = H_0 \oplus (\bigoplus_i l^2(Y_i)),
\]

the restriction \( C^*_\varphi(X) |_{H_0} \) is irreducible and the restriction \( C^*_\varphi(X) |_{l^2(Y_i)} \) is isomorphic to a \( C^* \)-algebra generated by mapping with minimal element.
The structure of $l^2(Y_i)$ depends on the structure of the sets \( \{ \varphi^{-m}(x) \} = \{ y \in X : \varphi^m(y) = x \} \). We consider two cases.
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$$e_x \omega e_y \quad \text{and} \quad \varphi(x) = \varphi(y)$$

and

$$\text{card } E(z) = 1$$

for all $n \in \mathbb{N}$ and $z \in \varphi^{-n}(x) \cup \varphi^{-n}(y)$.

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for all $n \in \mathbb{N}$ and $z \in \varphi^{-n}(x) \cup \varphi^{-n}(y)$.

Then the Hilbert space generated by the elements as 
\[ \frac{e_{x_i} - e_{y_i}}{\sqrt{2}} \]
is invariant for $C^*_\varphi(X)$ if for all $i$:

- $e_{x_i} \omega e_{y_i}$;
- $x_i \sim y_i$;
- $x \prec x_i$ and $y \prec y_i$. 

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Example of $C^*$-algebra generated by mapping
Representation of \( C^*_\varphi(X) \) into \( C^* \)-algebra generated by operator of weighted shift

**Theorem**

Let \( e_x \omega e_y \), \( \varphi(x) = \varphi(y) \) and \( E(z) = \varphi^{-1}(\varphi(z)) \) for all \( n \in \mathbb{N} \) and \( z \in \varphi^{-n}(x) \cup \varphi^{-n}(y) \). Then

\[
l^2(X) = H_0 \oplus \left( \bigoplus l^2(\mathbb{Z}_+) \right)
\]

and

\[
T\varphi = T_0 \oplus \left( \bigoplus_{i=0}^{\infty} (U^* T U^i) \right),
\]

where \( U \) is a shift operator and \( T \) is an operator of weighted shift.
Example of $C^*$-algebra generated by mapping
Corollary

Let \( \varphi \) be a mapping such that for every \( e_x \sim e_y \) it is true that \( e_x \sim e_y \). Then there exist a permutation group \( G \) and nontrivial unitary representation \( \pi : G \to B(l^2(X)) \) such that

\[
\pi(g)T_\varphi = T_\varphi \pi(g) \quad \text{for all} \quad g \in G.
\]
In this case for every basis element $e_x$ there is countable number of basis elements which satisfy the condition $\omega$. So we have the finite or countable number of classes of $\omega$-equivalent basis elements, denoted via $[e_x]_{\omega_i}$. 
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**Definition**

The class $[e_x]_{\omega_i}$ is principal if there is an $m$ such that

$$e_x \omega e_{\varphi^m(x)} \text{ for all } e_x \in [e_x]_{\omega_i}.$$  

The minimum of such $m$ we will call the period of principal class of equivalent basis elements.
Lemma
All principal classes of $\omega$-equivalent basis elements have the same period. The number of these classes is finite and coincides with $m$ — period of ones.
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Corollary
Let $\varphi$ be a mapping such that for every $e_x \omega e_y$ it is true that $e_x \sim e_y$. Then there exist a nontrivial unitary representation $\pi : \mathbb{Z} \rightarrow B(l^2(X))$ such that

$$\pi(n) T_\varphi = T_\varphi \pi(n) \quad \text{for all} \quad n \in \mathbb{Z},$$

and the classes $[e_x]_\omega$ are invariant under the action of $\pi$. 
Theorem

Let $\varphi$ be a mapping such that for every $e_x \sim e_y$, $e_x \sim e_y$ and there is the finite number of classes of $\omega$-equivalent basis elements $[e_x]_{\omega_i}$. Then

$$l^2(X) \simeq l^2(\mathbb{Z}) \otimes H$$

and

$$C^* \varphi(X) \simeq C(S^1, B),$$

where $B$ is a finite-dimensional algebra.
Example of $C^*_\varphi(X)$ with the only principal class of equivalent basis elements

\[ l^2(X) \cong l^2(\mathbb{Z}) \otimes H_2 \]

\[ C^*_\varphi(X) \cong C(S^1) \otimes M_2(\mathbb{C}) \]
Thank you!