Criterion of irreducibility of C^*-algebra generated by mapping

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Context of report

- C^*-algebra $C^*_\varphi(X)$ generated by mapping.
- Some properties of $C^*_\varphi(X)$.
- Criterion of irreducibility of the natural representation of $C^*_\varphi(X)$.
- Basic invariant subspaces of $C^*_\varphi(X)$.
- Some examples of $C^*_\varphi(X)$ illustrating the basic invariant subspaces.

Let X be an arbitrary countable set. Mapping $\varphi : X \longrightarrow X$ generates oriented graph (X, φ) with vertices in the elements of X and edges $(x, \varphi(x))$. We assume the cardinalities of preimages are bounded under the action of mapping φ.

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Mapping φ induces the mapping

$$T_\varphi : l^2(X) \longrightarrow l^2(X); \quad T_\varphi f = f \circ \varphi$$

and hence the family of partial isometries $\{U_k\}$ participating in decomposition of operator T_φ,

$$T_\varphi = U_1 + \sqrt{2} U_2 + \cdots + \sqrt{m} U_m + \cdots .$$

C^*-algebras generated by mappings

Definition

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The family of partial isometries satisfy the equalities:

$$U_1^* U_1 + U_2^* U_2 + \cdots + U_m^* U_m + \cdots = P;$$

$$U_1 U_1^* + U_2 U_2^* + \cdots + U_m U_m^* + \cdots = Q;$$

where P and Q are projections defined by the mapping φ.

We call an element from $\{U_k\}_{k=1}^{\infty} \cup \{U_k^*\}_{k=1}^{\infty}$ the *primary monomial*. We define

$$\operatorname{ind} U_k = -1 \quad \text{and} \quad \operatorname{ind} U_k^* = 1$$

We call V the *monomial* if it is any finite product of primary monomials not identically zero,

$$V = \prod_{k=1}^{m} U'_{j_k}, \quad U'_{j_k} \in \{U_{j_k} \cup U^*_{j_k}\}, \quad \text{and}$$

$$\operatorname{ind} V = \sum_{k=l}^{m} \operatorname{ind} U'_{j_k}.$$

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We assume X and φ to satisfy the following conditions:

- $\sup\limits_{y \in X} \mathrm{card}\varphi^{-1}(y) = m < \infty$;
- graph (X, φ) is connected;
- there is no element in X such that $\varphi^n(y) = y$ for some $n \in \mathbb{N}$.

Structure of $C^*_\varphi(X)$

In a case of absence of cyclic elements the index of monomial is well-defined. Hence we can equip $C^*_\varphi(X)$ with a circle action α.

$$C^*_\varphi(X) = \overline{\bigoplus_{n\in\mathbb{Z}} C^*_{\varphi,n}}.$$

The linear combinations of monomials of index n are dense in n^{th} spectral subspace

$$C^*_{\varphi,n} = \{A \in C^*_\varphi(X) : \alpha_z(A) = z^n A \quad \text{for} \quad z \in S^1\}.$$

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The fixed point subalgebra $C^*_{\varphi,0}$ (generated by monomials of index zero) is AF-algebra.

Criterion of irreducibility of the natural representation of $C^*_\varphi(X)$

Mapping φ induces the partial order in X. We will write

$$x \prec y \quad \text{if there is an} \quad m \in \mathbb{Z}_+ \quad \text{such that} \quad \varphi^m(y) = x$$

and

$$x \sim y \quad \text{if there is an} \quad m \in \mathbb{Z}_+ \quad \text{such that} \quad \varphi^m(x) = \varphi^m(y).$$

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$$e_x \prec e_y \quad \text{if} \quad x \prec y$$

and

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Theorem

Let (X, φ) be a connected graph, $\{e_x\}$ a natural basis in $l^2(X)$. Then the following are equivalent:

- $C^*_\varphi(X)$ *is irreducible on* $l^2(X)$*;*
- *if* $(Ve_x, e_x) = (Ve_y, e_y)$ *for all monomials* $V \in C^*_\varphi(X)$*, then* $x = y$.

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Definition

Let $\{e_x\}$ be a natural basis in $l^2(X)$. We will say that the basis elements e_x and e_y satisfy the condition ω ($e_x \, \omega \, e_y$) if $(Ve_x, e_x) = (Ve_y, e_y)$ for all $V \in C^*_\varphi(X)$.

$C^*_\varphi(X)$ is irreducible $\iff$ $e_x \, \omega \, e_x$ only for all $x \in X$.

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Let $e_x \, \omega \, e_y$. Then

- *for every $x' \in \varphi^{-1}(x)$ such an $y' \in \varphi^{-1}(y)$ exists that $e_{x'} \, \omega \, e_{y'}$;*
- $\operatorname{card} E(x') = \operatorname{card} E(y')$.

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- $e_x \,\omega\, e_y, \quad e_x \sim e_y$;
- $e_x \,\omega\, e_y, \quad e_x \nsim e_y$.

Let Y be a countable set with a mapping $\psi : Y \longrightarrow Y \cup \{\emptyset\}$ such that there exists the minimal element y_0 (for all $y \in Y$ there is an m such that $\varphi^m(y) = y_0$) and $\psi(y_0) = \emptyset$.

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Proposition

Let $\psi : Y \longrightarrow Y \cup \{\emptyset\}$ be a mapping with minimal element generating C^*-algebra $\mathfrak{A}_\psi$.
Then there exist a set X and mapping $\varphi : X \longrightarrow X$ such that:

- $l^2(Y) \hookrightarrow l^2(X)$;
- $\mathrm{Im}(l^2(Y))$ *is invariant for* $C^*_\varphi(X)$;
- $\Omega : C^*_\varphi(X) \longrightarrow \mathfrak{A}_\psi$ — *surjective* $*$*-homomorphism.*

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Proposition

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Theorem

Let φ be a mapping such that all $e_x \,\omega\, e_y$ are equivalent. Then

$$l^2(X) = H_0 \oplus (\bigoplus_i l^2(Y_i)),$$

the restriction $C^*_\varphi(X)_{|H_0}$ is irreducible and the restriction $C^*_\varphi(X)_{|l^2(Y_i)}$ is isomorphic to a C^*-algebra generated by mapping with minimal element.

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$$\operatorname{card} E(z) = 1$$

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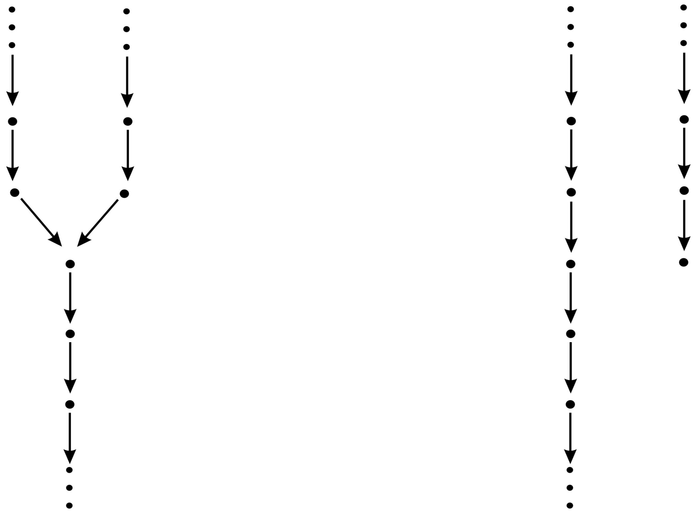
for all $n \in \mathbb{N}$ and $z \in \varphi^{-n}(x) \cup \varphi^{-n}(y)$.
Then the Hilbert space generated by the elements as

$$\frac{e_{x_i} - e_{y_i}}{\sqrt{2}}$$

is invariant for $C^*_\varphi(X)$ if for all i:

- $e_{x_i} \, \omega \, e_{y_i}$;
- $x_i \sim y_i$;
- $x \prec x_i$ and $y \prec y_i$.

Example of C^*-algebra generated by mapping



Representation of $C^*_\varphi(X)$ into C^*-algebra generated by operator of weighted shift

Theorem

Let $e_x \,\omega\, e_y$, $\varphi(x) = \varphi(y)$ and $E(z) = \varphi^{-1}(\varphi(z))$ for all $n \in \mathbb{N}$ and $z \in \varphi^{-n}(x) \cup \varphi^{-n}(y)$. Then

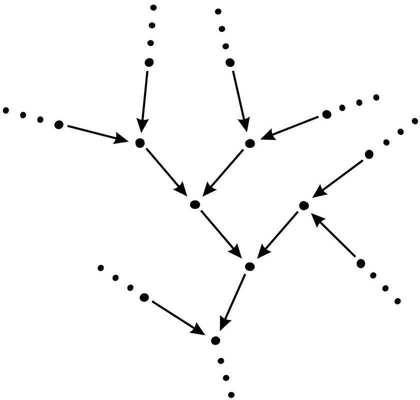
$$l^2(X) = H_0 \oplus \big(\oplus\, l^2(\mathbb{Z}_+)\big)$$

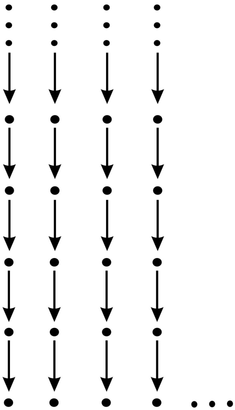
and

$$T_\varphi = T_0 \oplus \big(\bigoplus_{i=0}^{\infty} (U^{*i} T U^i)\big),$$

where U is a shift operator and T is an operator of weighted shift.

Example of C^*-algebra generated by mapping





Corollary

Let φ be a mapping such that for every $e_x\,\omega\,e_y$ it is true that $e_x \sim e_y$. Then there exist a permutation group G and nontrivial unitary representation $\pi : G \longrightarrow B(l^2(X))$ such that

$$\pi(g)T_\varphi = T_\varphi\pi(g) \quad \text{for all} \quad g \in G.$$

Case $e_x \, \omega \, e_y$, $\quad e_x \not\propto e_y$

In this case for every basis element e_x there is countable number of basis elements which satisfy the condition ω. So we have the finite or countable number of classes of ω-equivalent basis elements, denoted via $[e_x]_{\omega_i}$.

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Definition

The class $[e_x]_{\omega_i}$ is principal if there is an m such that

$$e_x \, \omega \, e_{\varphi^m(x)} \quad \textit{for all} \quad e_x \in [e_x]_{\omega_i}.$$

The minimum of such m we will call the period of principal class of equivalent basis elements.

Lemma

All principal classes of ω-equivalent basis elements have the same period. The number of these classes is finite and coincides with m — period of ones.

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Corollary

Let φ be a mapping such that for every $e_x \, \omega \, e_y$ it is true that $e_x \nsim e_y$. Then there exist a nontrivial unitary representation $\pi : \mathbb{Z} \longrightarrow B(l^2(X))$ such that

$$\pi(n) T_\varphi = T_\varphi \pi(n) \quad \text{for all} \quad n \in \mathbb{Z},$$

and the classes $[e_x]_{\omega_i}$ are invariant under the action of π.

Theorem

Let φ be a mapping such that for every $e_x \, \omega \, e_y$, $e_x \nsim e_y$ and there is the finite number of classes of ω-equivalent basis elements $[e_x]_{\omega_i}$. Then

$$l^2(X) \simeq l^2(\mathbb{Z}) \otimes H$$

and

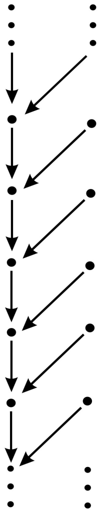
$$C^*\varphi(X) \simeq C(S^1, B),$$

where B is a finite-dimensional algebra.

Example of $C^*_\varphi(X)$ with the only principal class of equivalent basis elements

$$l^2(X) \simeq l^2(\mathbb{Z}) \otimes H_2$$

$$C^*_\varphi(X) \simeq C(S^1) \otimes M_2(\mathbb{C})$$



Thank you!