Operator amenability of the L^1-algebras of compact quantum groups

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Banach algebras 2013

Amenability of $L^1(G)$ for a locally compact group G

- A Banach algebra $\mathcal{A}$ is called **amenable** if every bounded derivations $D : \mathcal{A} \to X^*$ is inner for any $\mathcal{A}$-bimodule X.
- A locally compact group G is called **amenable** if there is a left invariant mean on $L^\infty(G)$ (i.e. there is a state m of $L^\infty(G)$ s.t. $m(\lambda(x)f) = m(f)$, $x \in G$, $f \in L^\infty(G)$, where $\lambda : G \to B(L^2(G))$, $\lambda(x)f(y) = f(x^{-1}y)$, $x, y \in G$, the left regular representation of G.)
- (**Johnson '72**) $L^1(G)$ is amenable $\Leftrightarrow$ G is amenable.

The Fourier algebra $A(G)$ and operator amenability

- Recall the group VN-alg. $VN(G) = \{\lambda(x)\}''_{x \in G} \subseteq B(L^2(G))$.
- The comultiplication $\Delta : VN(G) \to VN(G \times G),\ \lambda(x) \mapsto \lambda(x) \otimes \lambda(x)$ is a unital normal $*$-isomorphism, so that $A(G) = VN(G)_*$, the predual of $VN(G)$ can be equipped with a Banach algebra structure given by Δ_*. We call $A(G)$ the **Fourier algebra** on G.
- Note that if we identify $A(G) = \{f * \check{g} : f, g \in L^2(G)\}$, then Δ_* is nothing but the pointwise multiplication.
- (**Ruan '95**) $A(G)$ is **operator amenable** (i.e. amenable in the category of operator spaces) $\Leftrightarrow$ G is amenable.

Extension to quantum groups

- (**Def**) A compact quantum group $\mathbb{G}$ is given by (A, Δ), a unital C^*-algebra $A = C(\mathbb{G})$ and a unital $*$-isomorphism $\Delta : A \to A \otimes_{\text{min}} A$ called the **co-multiplication** with
  - **co-associativity** $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$
  - **cancellation law**

    $$\Delta(A)(A \otimes 1) = \text{span}\{\Delta(a)(b \otimes 1) : a, b \in A\} \text{ and } \Delta(A)(1 \otimes A)$$

    are dense in $A \otimes_{\text{min}} A$.
- For a compact quantum group $\mathbb{G} = (A, \Delta)$ there is a unique *Haar state* h on A such that

  $$(h \otimes id)\Delta(a) = h(a)1 = (id \otimes h)\Delta(a), \quad a \in A.$$

- $\mathbb{G}$ is called **Kac-type (or a Kac algebra)** is h is tracial.

Extension to quantum groups: continued

- The **reduced** version $C(\mathbb{G})_{\text{red}}$ of $C(\mathbb{G})$ is the C^*-algebra $\rho(C(\mathbb{G}))$ for the GNS representation ρ of h. We define $L^\infty(\mathbb{G}) := C(\mathbb{G})''_{\text{red}}$. Then the co-multiplication $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes_{\min} C(\mathbb{G})$ can be naturally extended to a unital normal $*$-isomorphism $\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\mathbb{G})$.
- The pre-adjoint map $\Delta_* : L^1(\mathbb{G}) \widehat{\otimes} L^1(\mathbb{G}) \to L^1(\mathbb{G})$, where $L^1(\mathbb{G}) = L^\infty(\mathbb{G})_*$ gives us a (completely contractive) Banach algebra structure on $L^1(\mathbb{G})$.
- Note that $(L^\infty(\mathbb{G}), \Delta, h)$ is a VN-algebraic compact quantum group in the Kusterman-Vaes sense.

Extension to quantum groups: continued 2

- Can we extend the connection between operator amenability of $L^1(\mathbb{G})$ and amenability of $\mathbb{G}$ to the quantum case?
- (**Ruan '96**) Let $\mathbb{G}$ be a compact Kac algebra. Then, $L^1(\mathbb{G})$ is operator amenable iff $\mathbb{G}$ is co-amenable (i.e. $L^1(\mathbb{G})$ has a bounded approximate identity (shortly BAI)).
- (**Easy implications**)
  $\mathbb{G}$: a locally compact quantum group
  $L^1(\mathbb{G})$ is operator amenable $\Rightarrow$ $\mathbb{G}$ is amenable (i.e. having a left invariant mean on $L^\infty(\mathbb{G})$) and co-amenable.
- (**Conjecture by V. Runde**) We will actually have equivalence in the above.
- We will answer the above conjecture in a negative way.
- Note that a compact quantum group $\mathbb{G}$ is always amenable. Actually, the Haar state does the job.

Biprojectivity and Biflatness

- $\mathcal{A}$: completely contractive Banach algebra with the algebra multiplication $m : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$:
  $\mathcal{A}$ is called **operator biprojective** if $\exists$ a cb-$\mathcal{A}$-bimodule map $\rho : \mathcal{A} \to \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $m \circ \rho = id_{\mathcal{A}}$.
  $\mathcal{A}$ is called **operator biflat** if $\exists$ a cb-$\mathcal{A}$-bimodule map $\theta : (\mathcal{A}\widehat{\otimes}\mathcal{A})^* \to \mathcal{A}^*$ such that $\theta \circ m^* = id_{\mathcal{A}^*}$.
- In the above cases ρ and θ are called **splitting homomorphisms**.
- (**Facts**)
  Operator biprojectivity $\Rightarrow$ operator biflatness.
  Operator amenability $\Leftrightarrow$ operator biflatness + having a BAI.

Biprojectivity and Biflatness: continued

- G : a locally compact group.
  (**Helemskii, '85**)
  $L^1(G)$ is operator biprojective $\Leftrightarrow$ G is compact.
  (**Aristov, '02, Wood, '02**)
  $A(G)$ operator biprojective $\Leftrightarrow$ G is discrete.
- $L^1(G)$ is operator biflat $\Leftrightarrow$ G is amenable.
  (**Aristov/Runde/Spronk, '04**)
  $A(G)$ operator biflat $\Leftarrow$ G is a quasi-[SIN] group (i.e. there is a BAI (e_α) in $L^1(G)$ s.t. $\delta_x * e_\alpha - e_\alpha * \delta_x \to 0$, $x \in G$), a strictly bigger class than amenable groups and discrete groups.

Biprojectivity and Biflatness: quantum cases

- (**Aristov '04, Daws '10, Ruan/Xu '96**) Some general theory
  - $\mathbb{G}$: LCQGp with $L^1(\mathbb{G})$ operator biprojective $\Rightarrow$ $\mathbb{G}$ is compact.
  - $\mathbb{G}$: compact Kac algebra $\Rightarrow$ $L^1(\mathbb{G})$ is operator biprojective.
  - $L^1(\mathbb{G})$ is operator biprojective with completely contractive (or completely positive) splitting hom. $\Rightarrow$ $\mathbb{G}$ is of Kac-type.
- (**Question by Aristov**) Is $L^1(\mathbb{G})$ operator biprojective for any compact quantum group $\mathbb{G}$?

The main results

Theorem

Let $\mathbb{G}$ be a compact quantum group. If $L^1(\mathbb{G})$ is operator biflat, then $\mathbb{G}$ is of Kac type.

Corollary

Let $\mathbb{G}$ be a compact quantum group. Then, $L^1(\mathbb{G})$ is operator amenable if and only if $\mathbb{G}$ is co-amenable and of Kac type.

Note that there are many examples of co-amenable compact quantum group of non-Kac type, e.g. $SU_q(2)$, $0 < q < 1$.

Corollary

Let $\mathbb{G}$ be a locally compact quantum group. Then, $L^1(\mathbb{G})$ is operator biprojective if and only if $\mathbb{G}$ is compact and of Kac type.

More on compact quantum groups

- A *finite dimensional corepresentation* of $\mathbb{G}$ is a matrix $u = (u_{ij}) \in M_n(A)$ such that

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \;\; 1 \leq i,j \leq n.$$

  We say that the corepresentation u is unitary if u is a unitary matrix and irreducible if we have $\{X \in M_n : Xu = uX\} = \mathbb{C}1$. The number n is called the *dimension* of u.

- Let $\{u^\alpha : \alpha \in \mathcal{I}\}$ be a maximal family of finite dimensional irreducible unitary corepresentations of $\mathbb{G}$.

An example: $SU_q(2)$, $0 < q < 1$

- $C(SU_q(2)) :=$ the universal C^*-algebra generated by α, γ s.t.

  $\alpha^*\alpha + \gamma^*\gamma = \alpha\alpha^* + q^2\gamma\gamma^* = 1,\ \gamma\gamma^* = \gamma^*\gamma,\ q\gamma\alpha = \alpha\gamma,\ q\gamma^*\alpha = \alpha\gamma^*.$

- The co-multiplication is given by

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma,\ \ \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

- There is a concrete realization of α and γ as follows:

$$\alpha(e_i \otimes f_k) = \sqrt{1 - q^{2i}}\, e_{i-1} \otimes f_k,\ \ \gamma(e_i \otimes f_k) = q^i e_i \otimes f_{k+1}$$

  where e_i and f_k are canonical ONB on $\ell^2(\mathbb{N})$ and $L^2(\mathbb{T})$, respectively.

An example: $SU_q(2)$, $0 < q < 1$: continued

- The co-representation theory of $SU_q(2)$ is well-understood. The index set is $\mathcal{I} = \frac{1}{2}\mathbb{Z}^+$, and for $l \in \mathcal{I}$ and $m = -l, -l+1, \cdots, l$ we have $u^0 = 1$, $u^{\frac{1}{2}} = \begin{bmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{bmatrix}$,

$$u^l_{m,l} = \binom{2l}{l-m}^{\frac{1}{2}}_{q^{-2}} (-q\gamma^*)^{l-m}(\alpha^*)^{l+m} \text{ and}$$

$$u^l_{l,m} = \binom{2l}{l-m}^{\frac{1}{2}}_{q^{-2}} \gamma^{l-m}(\alpha^*)^{l+m}.$$

- $SU_q(2)$ is known to be co-amenable and of non-Kac type.

Sketch of the proof for $SU_q(2)$

- (**A modification of M. Daws' observation**)
  Suppose that $L^1(\mathbb{G})$ is operator biflat with the splitting homomorphism $\theta : L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$. Then, there exists a family $\{X^\alpha \in M_{n_\alpha} : \alpha \in \mathcal{I}\}$ such that for $\alpha, \beta \in \mathcal{I}$, $1 \le i, j \le n_\alpha$ and $1 \le k, l \le b_\beta$,

$$\theta(u^\alpha_{ij} \otimes u^\beta_{kl}) = \delta_{\alpha\beta} X^\alpha_{jk} u^\alpha_{il}, \;\; \sum_{r=1}^{n_\alpha} X^\alpha_{rr} = 1. \tag{0.1}$$

Sketch of the proof for $SU_q(2)$: continued

- Let θ be a splitting homomorphism. Then for $l \in \mathcal{I}$ and half-integers $-l \leq n, m, n', m' \leq l$ we have

$$\theta(u^l_{n,m} \otimes u^l_{n',m'}) = X^l_{m,n'} u^l_{n,m'}.$$

Since $\sum_{m=-l}^{l} X^l_{m,m} = 1$ there is $-l \leq m \leq l$ such that

$$\left| X^l_{m.m} \right| \geq \frac{1}{2l+1}.$$

- Now we see that $u^l_{m,l} = (-q)^{l-m} \Phi(u^{(l)}_{l,m})$, where $\Phi : C(SU_q(2)) \to C(SU_q(2)),\ \alpha \mapsto \alpha,\ \gamma \mapsto \gamma^*$, a $*$-isomorphism. Indeed, (α, γ^*) satisfy the same set of relations. Thus, we have $\left\| u^l_{m,l} \right\| = |q|^{l-m} \left\| u^l_{l,m} \right\| \leq |q|^{l-m}$. Similarly, we have $\left\| u^l_{-l,m} \right\| \leq |q|^{l+m}$.

Sketch of the proof for $SU_q(2)$: continued 2

- Moreover, $u^l_{l,l} = (\alpha^*)^{2l}$ and $u^l_{-l,-l} = \alpha^{2l}$, so that we have $\left\| u^l_{l,l} \right\| = \left\| u^l_{-l,-l} \right\| = 1$ from the concrete realization of α.
- Combining all the above estimates we get

$$\|\theta\| \geq \frac{\left\| X^l_{m,m} u^l_{l,l} \right\|}{\left\| u^l_{l,m} \otimes u^l_{m,l} \right\|} \geq \frac{|q|^{-l+m}}{2l+1}$$

and $\|\theta\| \geq \frac{|q|^{-l-m}}{2l+1}$. Consequently, by taking the geometric mean we get

$$\|\theta\| \geq \frac{|q|^{-l}}{2l+1},$$

which is contradictory to boundedness of θ.

Remark

- We actually showed that $L^1(SU_q(2))$ is not biflat (in the Banach space category). This is not surprising in this case since $SU_q(2)$ is co-amenable. However, we can apply the same techique to show that $L^1(A_o(F))$ is not biflat for some free orthogonal quantum groups $A_o(F)$ which is not co-amenable.

Questions

- Is $L^1(SU_q(2))$ operator weakly amenable?
- Is $L^1(\widehat{SU_q(2)})$ operator amenable, equivalently, operator biflat?
- The case of non-compact and non-discrete quantum groups.