An almost $C_0(\mathcal{K})$-C*-algebra

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Let $\mathfrak{g}_6$ be a Lie algebra spanned by the vectors A, B, P, Q, R, S and equipped with the Lie brackets:

$$[P,Q]=R,\qquad [P,R]=S,\qquad [A,P]=\frac{1}{2}P,\qquad [B,P]=-\frac{1}{2}P,$$

$$[B,Q]=Q,\qquad [A,R]=\frac{1}{2}R,\qquad [B,R]=\frac{1}{2}R,\qquad [A,S]=S,$$

$$[A,Q]=0,\qquad [B,S]=0.$$

The group $G_6 := \exp\mathfrak{g}_6$ is a 6-dimensional exponential Lie group.

Question: What is the $C^*(G_6)$?

Group C*-algebras

Let G be a locally compact group, dx be the Haar measure on G. The Banach space $L^1(G)$ is of integrable functions on G with respect to dx with the convolution product $*$, i.e.

$$f * g(y) := \int_G f(x)g(x^{-1}y)dx \text{ for } f, g \in L^1(G) \text{ and } y \in G.$$

$\Longrightarrow (L^1(G), *)$ is a Banach algebra which has an isometric involution defined by

$$f^*(x) := \Delta_G(x)^{-1}\overline{f(x^{-1})}, f \in L^1(G),\ x \in G,$$

where Δ_G denotes the modular function on G.

Representations

Every irreducible unitary representation $(\pi, \mathcal{H}_\pi)$ of G integrates to an irreducible representation (also denoted by π) of $L^1(G)$ in the sense:

$$\pi(F) = \int_G F(x)\pi(x)dx, \ F \in L^1(G),$$

the integral converges in the Banach space $\mathcal{B}(\mathcal{H}_\pi)$.
Furthermore, for every irreducible representation $\overline{\pi}$ of the algebra $L^1(G)$ on a Hilbert space $\mathcal{H}$, there exists a unique irreducible unitary representation $(\pi, \mathcal{H})$ of G such that $\overline{\pi}(F) = \pi(F)$ for $F \in L^1(G)$.

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The C*-algebra of G:

$$C^*(G) := \overline{L^1(G)}^{\|\cdot\|_{C^*}}$$

with the C*-norm

$$\|F\|_{C^*} := \sup_{\pi \in \widehat{G}} \|\pi(F)\|_{\text{op}}.$$

The unitary dual space $\widehat{C^*(G)}$ of the $C^*(G)$ can be identified with the unitary dual $\widehat{G}$ of G.

Fourier transform

To analyze a C*-algebra $\mathcal{A}$, one can use the Fourier transform $\mathcal{F}$ to decompose $\mathcal{A}$ over its unitary dual $\widehat{\mathcal{A}}$.

To be able to define this transform, consider the algebra $\ell^\infty(\widehat{\mathcal{A}})$ of bounded operator fields defined over $\widehat{\mathcal{A}}$ by

$$\begin{aligned}\ell^\infty(\widehat{\mathcal{A}}) \; := \; & \{A = (A(\pi) \in \mathcal{B}(\mathcal{H}_\pi))_{\pi \in \widehat{\mathcal{A}}}, \\ & \|A\|_\infty := \sup_\pi \|A(\pi)\|_{\mathrm{op}} < \infty\}.\end{aligned}$$

The space $\ell^\infty(\widehat{\mathcal{A}})$ is a C*-algebra itself. The Fourier transform $\mathcal{F}$ defined by

$$\mathcal{F}(a) := \hat{a} := (\pi(a))_{\pi \in \widehat{\mathcal{A}}} \quad \text{for} \quad a \in \mathcal{A}$$

is then an injective, hence isometric, homomorphism of $\mathcal{A}$ into $\ell^\infty(\widehat{\mathcal{A}})$.

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Let A be a separable C*-algebra, $\widehat{A}$ be its unitary dual. Suppose there are finitely many closed subsets $S_0 \subset S_1 \subset \ldots \subset S_d$ in $\widehat{A}$ s.t. $\Gamma_0 := S_0$ and $\Gamma_i := S_i \setminus S_{i-1}$ are Hausdorff.

Definition

For $S \subset \widehat{A}$ closed, denote by $CB(S)$ the *-algebra of all uniformly bounded operator fields $(\psi(\gamma) \in \mathcal{B}(\mathcal{H}_i))_{\gamma \in S \cap \Gamma_i, i=1,\cdots,d}$, which are operator norm continuous on the subsets $\Gamma_i \cap S$ for every $i = 1, \ldots, d$ for which $\Gamma_i \cap S \neq \emptyset$. We provide the algebra $CB(S)$ with the norm,

$$\|\psi\|_S := \sup_{\gamma \in S} \|\psi(\gamma)\|_{\mathrm{op}}.$$

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Almost $C_0(\mathcal{K})$-C*-algebras

Definition

Let A be a C*-algebra satisfying the conditions:

1. The mappings $\gamma \mapsto \mathcal{F}(a)(\gamma)$ are norm continuous on each Γ_i.
2. For each $i = 1, \cdots, d$, there is a sequence $(\sigma_{i,k} : CB(S_{i-1}) \to CB(S_i))_k$ of linear mappings which are uniformly bounded in k such that

$$\lim_{k\to\infty} \mathrm{dis}\Big((\sigma_{i,k}(\mathcal{F}(a)_{|S_{i-1}}) - \mathcal{F}(a)_{|\Gamma_i}), C_0(\Gamma_i, \mathcal{K}(\mathcal{H}_i))\Big) = 0,$$

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We call such a C*-algebra *almost $C_0(\mathcal{K})$*.

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Let A be a separable C*-algebra which is almost $C_0(\mathcal{K})$. For every $a \in A$, the operator field $\mathcal{F}(a)$ defined over $\widehat{A}$ is in $\ell^\infty(\widehat{A})$ with

1. The field $\mathcal{F}(a)$ is uniformly bounded, i.e.,

$$\|\mathcal{F}(a)\| := \sup_{\gamma \in \widehat{A}} \|\mathcal{F}(a)(\gamma)\|_{\mathrm{op}} < \infty.$$

2. $\mathcal{F}(a)_{|\Gamma_i} \in CB(\Gamma_i)$, for every $i = 1, \ldots, d$.
3. For every $(\gamma_k)_{k\in\mathbb{N}}$ in $\widehat{A}$ going to infinity,

$$\lim_{k\to\infty} \|\mathcal{F}(a)(\gamma_k)\|_{\mathrm{op}} = 0.$$

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A C*-subalgebra of $\ell^\infty(\widehat{A})$

Let $D^*(A)$ be the set of all operator fields defined over $\widehat{A}$ satisfying the above 4 conditions.

Theorem

Let A be a separable C-algebra which is almost $C_0(\mathcal{K})$. Then the subset $D^*(A)$ of $\ell^\infty(\widehat{A})$ is a C*-subalgebra of $\ell^\infty(\widehat{A})$ which is isomorphic to A under the Fourier transform.*

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$$[A, Q] = 0, \qquad [B, S] = 0.$$

The group $G_6 := \exp \mathfrak{g}_6$ is a 6-dimensional exponential Lie group. Let $\{A^*, B^*, P^*, Q^*, R^*, S^*\}$ be the dual basis of $\{A, B, P, Q, R, S\}$. We have the following parametrization of the coadjoint orbits $\mathfrak{g}_6^*/\operatorname{Ad}^* G_6$.

0-dimensional orbits $a^*A^* + b^*B^*$, where $a^*, b^* \in \mathbb{R}$.

2-dimensional orbits

(2d-1) $\alpha^*(\frac{A^*+B^*}{2}) + \varepsilon P^*$, where $\alpha^* \in \mathbb{R}$, $\varepsilon = \pm 1$.
(2d-2) $a^*A^* + \delta Q^*$, where $a^* \in \mathbb{R}$, $\delta = \pm 1$.

4-dimensional orbits

(4d-1) $\varepsilon P^* + \delta Q^*$, where $\varepsilon = \pm 1, \delta = \pm 1$.
(4d-2) $b^*B^* + \varepsilon R^*$, where $b^* \in \mathbb{R}$, $\varepsilon = \pm 1$.
(4d-3) $b^*B^* + \varepsilon S^*$, where $b^* \in \mathbb{R}$, $\varepsilon = \pm 1$.

6-dimensional orbits

(6d) $\delta Q^* + \varepsilon S^*$, where $\delta = \pm 1$, $\varepsilon = \pm 1$.

The continuity condition

Let

- $\Gamma_0 := \{a^*A^* + b^*B^*, a^*, b^* \in \mathbb{R}\}$,
- $\Gamma_1 := \{x^*X^* + \mathrm{Ad}^*(G_6)(\varepsilon P^*),\ x^* \in \mathbb{R}^+, \varepsilon = \pm 1\}$,
- $\Gamma_2 := \{a^*A^* + \mathrm{Ad}^*(G_6)(\nu Q^*),\ a^* \in \mathbb{R}^+, \nu = \pm 1\}$,
- $\Gamma_3 := \{\Omega_{\varepsilon P^* + \nu Q^*}, \nu, \varepsilon = \pm 1\}$,
- $\Gamma_4 := \{b^*B^* + \mathrm{Ad}^*(G_6)(\varepsilon R^*), b^* \in \mathbb{R}^+, \varepsilon = \pm 1\}$,
- $\Gamma_5 := \{b^*B^* + \mathrm{Ad}^*(G_6)(\varepsilon S^*), b^* \in \mathbb{R}^+, \varepsilon = \pm 1\}$,
- $\Gamma_6 := \{\Omega_{\varepsilon S^* + \nu Q^*}, \varepsilon, \nu = \pm 1\}$.

Let $S_i := \bigcup_{j=0}^{i} \Gamma_j \subset \mathfrak{g}_6^*/G_6 \cong \widehat{G_6}$. Then S_i is closed in S_{i+1} for $i = 0, \ldots, 5$, the set Γ_6 is finite and open in $\widehat{G}_6$, and Γ_3 is finite and open in S_3. The sets $\Gamma_1, \Gamma_2, \Gamma_4$ and Γ_5 are homeomorphic to two disjoint copies of the real line $\mathbb{R}$.

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Theorem (The continuity condition)

The mappings $\pi \mapsto \pi(F)$ $(F \in C^*(G_6))$ *are norm continuous on the subsets* Γ_i *of* $\widehat{G_6}$ *for* $i = 0, \ldots, 6$.

Let $D^*(G_6)$ be the subset of $\ell^\infty(\widehat{G_6})$ consisting of all operator fields ϕ such that $\gamma \mapsto \phi(\gamma)$ is norm continuous and vanishes at infinity on each $\Gamma_1, \Gamma_2, \Gamma_4, \Gamma_5$ and for $\varepsilon, \omega = \pm 1$,

1. $\lim_{\delta\to 0} \mathrm{dis}(\phi(\varepsilon(S^* + Q^*)) - \sigma_{\varepsilon S^* + \varepsilon Q^*, \delta}(\phi), \mathcal{K}(L^2(\mathbb{R}^3))) = 0.$
2. $\lim_{\delta\to 0} \mathrm{dis}((\phi(\varepsilon(S^* - Q^*)) - \sigma_{\varepsilon S^* - \varepsilon Q^*, \delta}(\phi), \mathcal{K}(L^2(\mathbb{R}^3))) = 0.$
3. $\lim_{\delta\to 0} \mathrm{dis}(\tau_{\varepsilon S^*}(\phi) - \rho_{\varepsilon S^*}(\sigma_{\varepsilon S^*, \delta, D}(\phi)), C_0(\mathbb{R}, \mathcal{K})) = 0.$
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7. $\lim_{\delta\to 0} \mathrm{dis}(\tau_{\nu Q^*}(\phi) - \rho_{\nu Q^*}(\sigma_{\nu Q^*, \delta}(\phi))), C_0(\mathbb{R}, \mathcal{K})) = 0.$
8. The same conditions hold for the adjoint field ϕ^*.

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3. $\lim_{\delta\to 0} \operatorname{dis}(\tau_{\varepsilon S^*}(\phi) - \rho_{\varepsilon S^*}(\sigma_{\varepsilon S^*, \delta, D}(\phi)), C_0(\mathbb{R}, \mathcal{K})) = 0.$
4. $\lim_{\delta\to 0} \operatorname{dis}(\tau_{\varepsilon R^*}(\phi) - \rho_{\varepsilon R^*}(\sigma_{\varepsilon R^*, \delta, D}(\phi)), C_0(\mathbb{R}, \mathcal{K})) = 0.$
5. $\lim_{\delta\to 0} \operatorname{dis}((\phi(\varepsilon P^* + \nu Q^*) - \sigma_{\varepsilon P^* + \nu Q^*, \delta}(\phi)), \mathcal{K}(L^2(\mathbb{R}^3))) = 0.$
6. $\lim_{\delta\to 0} \operatorname{dis}(\tau_{\varepsilon P^*}(\phi) - \rho_{\varepsilon P^*}(\sigma_{\varepsilon P^*, \delta}(\phi)), C_0(\mathbb{R}, \mathcal{K})) = 0.$
7. $\lim_{\delta\to 0} \operatorname{dis}(\tau_{\nu Q^*}(\phi) - \rho_{\nu Q^*}(\sigma_{\nu Q^*, \delta}(\phi))), C_0(\mathbb{R}, \mathcal{K})) = 0.$
8. The same conditions hold for the adjoint field ϕ^*.

Theorem

The C-algebra of the group G_6 is almost $C_0(\mathcal{K})$. In fact, the Fourier transform maps $C^*(G_6)$ onto $D^*(G_6)$.*