Spectra of weighted composition operators with automorphic symbols

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We denote by $H(\mathbb{D})$ the space of analytic functions on the open unit disc $\mathbb{D}$ of the complex plane. For φ an analytic selfmap of $\mathbb{D}$ and u in $H(\mathbb{D})$, the ***weighted composition operator*** uC_φ is defined by

$$(uC_\varphi)f(z) = u(z)f(\varphi(z)),$$

where $f \in H(\mathbb{D})$.

There are two particularly interesting special cases of such operators: on one hand, taking $u = 1$ gives the ***composition operator*** C_φ, and on the other, putting $\varphi = id$, the identity function on $\mathbb{D}$, gives us the ***multiplication operator*** M_u.

The study of composition operators on analytic function spaces was started about 40 years ago by ***Nordgren, Kamowitz, Shapiro, Cowen and MacCluer*** and others.

There are at least two main goals by studying weighted composition operators.

• Relate properties of the functions u, φ to properties of the operators C_φ and uC_φ.

• Relate properties of the operators C_φ and uC_φ to properties of other operators, in order to better understand other operators.

For invertible uC_φ, we aim to

• calculate the **spectral radius** of uC_φ; and

• determine its **spectrum**.

The **methods of proof** should be general, so that the results hold on many large spaces of analytic functions. In fact, we have designed a unified approach to determine the spectra of invertible weighted composition operators on a broad class $\mathcal{A}$ of analytic function spaces.

The spectrum of uC_φ was studied by ***Gunatillake*** (2011) on the Hardy-Hilbert space $H^2(\mathbb{D})$ and by ***Kamowitz*** (1978) on the disc algebra $A(\mathbb{D})$.

Spectral properties of uC_φ depend on the type of the symbol φ. These types are:

- ***elliptic*** - φ has fixed point in $\mathbb{D}$;
- ***parabolic*** - φ has a unique fixed point in $\partial\mathbb{D}$;
- ***hyperbolic*** - φ has two fixed point in $\partial\mathbb{D}$.

The hyperbolic case is the most interesting one.

Fact (1987, Nordgren, Rosenthal, Wintrobe) : every linear bounded operator T has a closed non-trivial invariant subspace $\Leftrightarrow$ the minimal non-trivial closed invariant subspaces for C_φ on $H^2(\mathbb{D})$ are one-dimensional, where φ is a hyperbolic automorphism of $\mathbb{D}$.

Let $\mathcal{A} \subset H(\mathbb{D})$ be a Banach space; its norm is denoted by $||\cdot||_{\mathcal{A}}$. Suppose there is a constant $s > 0$ such that the following hold:

$1)$ For each $f \in \mathcal{A}$ and $z \in D$ we have

$$|f(z)| \lesssim ||f||_{\mathcal{A}}(1 - |z|^2)^{-s}.$$

$2)$ For each $z \in \mathbb{D}$ there is some $f_z \in \mathcal{A}$ with $||f_z||_{\mathcal{A}} \leq 1$ such that $f_z(z)(1 - |z|^2)^s = 1.$

$3)$ For φ an automorphism, $||C_\varphi|| \lesssim (1 - |\varphi(0)|^2)^{-s}.$

$4)$ For $w \in H^\infty(\mathbb{D})$ we have $||M_w|| \leq ||w||_\infty.$

The following spaces satisfy all the stated conditions.

• **Hardy spaces** $H^p(\mathbb{D})$, $1 \le p < \infty$, with $s = \frac{1}{p}$.

• **Weighted Bergman spaces** $A^p_\alpha(\mathbb{D})$, $p \ge 1$, $\alpha > -1$, consisting of all analytic functions f on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f(z)|^p (1 - |z|)^\alpha dA(z) < \infty.$$

Here $s = (\alpha + 2)/p$.

• The **weighted Banach spaces of analytic functions** $H^\infty_p(\mathbb{D})$, $0 < p < \infty$, consisting of analytic functions f on $\mathbb{D}$ with

$$\sup_{z \in \mathbb{D}} |f(z)|(1 - |z|)^p < \infty.$$

Here $s = p$.

Theorem 1. *For such $\mathcal{A}$ and φ an analytic selfmap of $\mathbb{D}$, the weighted composition operator uC_φ is Fredholm on $\mathcal{A}$ if and only if M_u is Fredholm and φ is an automorphism of the unit disc.*

In the rest of the talk we will assume that $uC_\varphi : \mathcal{A} \to \mathcal{A}$ is invertible and therefore φ is an automorphism. We have:

Theorem 2. *For such $\mathcal{A}$ and an automorphism φ the operator $uC_\varphi : \mathcal{A} \to \mathcal{A}$ is invertible if and only if u is bounded and bounded away from zero. The inverse is given by*

$$(uC_\varphi)^{-1} = \frac{1}{u \circ \varphi^{-1}} C_{\varphi^{-1}}.$$

Concerning the boundary fixed points of φ, one result of particular importance is the celebrated Denjoy-Wolff theorem.

If φ is either a ***parabolic*** or a ***hyperbolic automorphism*** of $\mathbb{D}$, this theorem guarantees that there is a (unique) fixed point $a \in \partial\mathbb{D}$ such that

$$\lim_{n\to\infty} \varphi_n(z) = a$$

uniformly on compact subsets of $\mathbb{D}$; the point a is called the ***Denjoy-Wolff point*** of φ. Moreover,

• if φ is parabolic, then $\varphi'(a) = 1$; and

• if φ is hyperbolic and its other fixed point $b \in \partial\mathbb{D}$, then $0 < \varphi'(a) < 1$ and $\varphi'(a) = 1/\varphi'(b)$.

The parabolic case

Theorem 3. *Let $\mathcal{A}$ be any of the spaces $A^p_\alpha(\mathbb{D})$, $p \geq 1$, $\alpha > -1$, and $s = \frac{\alpha+2}{p}$; $H^p(\mathbb{D})$, $p \geq 1$, and $s = \frac{1}{p}$; or $H^\infty_p(\mathbb{D})$, $0 < p < \infty$, and $s = p$. Suppose that $u \in A(\mathbb{D})$ is bounded away from zero on $\mathbb{D}$ and let φ be a parabolic automorphism of $\mathbb{D}$ whose Denjoy-Wolff point is $a \in \partial\mathbb{D}$. Then the spectral radius*

$$r(uC_\varphi) = |u(a)|$$

and the spectrum $\sigma_{\mathcal{A}}(uC_\varphi)$ is the circle

$$\sigma_{\mathcal{A}}(uC_\varphi) = \{\lambda \in \mathbb{C}; |\lambda| = |u(a)|\}.$$

The hyperbolic case

Theorem 4. *Let $\mathcal{A}$ be any of the spaces $A^p_\alpha(\mathbb{D})$, $p \geq 1$, $\alpha > -1$, and $s = \frac{\alpha+2}{p}$; $H^p(\mathbb{D})$, $p \geq 1$, and $s = \frac{1}{p}$; or $H^\infty_p(\mathbb{D})$, $0 < p < \infty$, and $s = p$. Let φ be a hyperbolic automorphism of $\mathbb{D}$ with fixed points a (attractive) and b (repulsive) in $\partial\mathbb{D}$. If $u \in A(\mathbb{D})$ is bounded away from zero, then*

$$r(uC_\varphi) = \max\left\{\frac{|u(b)|}{\varphi'(b)^s}, \frac{|u(a)|}{\varphi'(a)^s}\right\}$$

and if, moreover, $|u(b)/\varphi'(b)^s| \leq |u(a)/\varphi'(a)^s|$, then

$$\sigma_{\mathcal{A}}(uC_\varphi) = \left\{\lambda \in \mathbb{C}; \frac{|u(b)|}{\varphi'(b)^s} \leq |\lambda| \leq \frac{|u(a)|}{\varphi'(a)^s}\right\}.$$

The elliptic case

Theorem 5. *Let* $\mathcal{A}$ *be any of the spaces* $A^p_\alpha(\mathbb{D})$, $p \geq 1$, $\alpha > -1$, *and* $s = \frac{\alpha+2}{p}$; $H^p(\mathbb{D})$, $p \geq 1$, *and* $s = \frac{1}{p}$; *or* $H^\infty_p(\mathbb{D})$, $0 < p < \infty$, *and* $s = p$.

(a) *Suppose that* $u \in A(\mathbb{D})$ *and* φ *is an automorphism of* $\mathbb{D}$ *such that there is a positive integer* j *with* $\varphi_j(z) = z$ *for all* $z \in \mathbb{D}$*. If* n *is the smallest such integer, then*

$$\sigma_{\mathcal{A}}(uC_\varphi) = \overline{\{\lambda \in \mathbb{C}; \lambda^n = u_{(n)}(z), z \in \mathbb{D}\}},$$

where $u_{(n)} = \prod_{m=0}^{n-1} u \circ \varphi_m \in H(\mathbb{D})$.

(b) *Suppose* $u \in A(\mathbb{D})$ *is bounded away from zero on* $\mathbb{D}$ *and let* φ *be an elliptic automorphism such that* $\varphi_n(z) \not\equiv z$ *for all positive integers* n*. If* $a \in \mathbb{D}$ *is the unique fixed point of* φ*, then*

$$\sigma_{\mathcal{A}}(uC_\varphi) = \{\lambda \in \mathbb{C}; |\lambda| = |u(a)|\}.$$

Let us now consider two ***examples*** in which we determine the spectrum of a weighted composition operator uC_φ.

(1) Let uC_φ be a unitary weighted composition operator on $H^2(\mathbb{D})$, where φ is an automorphism. Then $s = 1/2$. Since uC_φ is unitary, it can be shown that

$$u(z) = c\frac{\sqrt{1-|z_0|^2}}{1-\overline{z_0}z},$$

where $\varphi(z_0) = 0$ and $|c| = 1$. Moreover, $r(uC_\varphi) = 1$ and $\sigma(uC_\varphi) \subseteq \partial\mathbb{D}$.

(a) Suppose that $\varphi(z) = \mu z$, where $|\mu| = 1$. Then $u(z) = c$. If $\mu^j = 1$ for some positive integer j, then if n is the smallest such integer, the previous Theorem 5 gives

$$\sigma_{H^2(\mathbb{D})}(uC_\varphi) = \overline{\{\lambda; \lambda^n = c^n\}} = \{\mu^k c; k = 0, 1, \ldots, n-1\}.$$

If there is no such integer, we get $\sigma_{H^2(\mathbb{D})}(uC_\varphi) = \{\lambda; |\lambda| = 1\}$.

(b) Let $\varphi(z) = \frac{(1+i)z-1}{z+i-1}$, that is, φ is a parabolic automorphism of $\mathbb{D}$. Then $z_0 = \frac{1}{1+i}$. By Theorem 3 $r(uC_\varphi) = |u(1)| = 1$ and the spectrum

$$\sigma_{H^2(\mathbb{D})}(uC_\varphi) = \{\lambda; |\lambda| = 1\}.$$

(c) Consider the hyperbolic automorphism $\varphi(z) = \frac{z+r}{1+rz}$ where $0 < r < 1$, that is, φ is a hyperbolic automorphism of $\mathbb{D}$ with Denjoy-Wolff point $a = 1$, and the other fixed point $b = -1$. Then $z_0 = -r$. In this case $|u(-1)| > |u(1)|$, and

$$\frac{|u(1)|}{\varphi'(1)^{1/2}} = \frac{|u(-1)|}{\varphi'(-1)^{1/2}} = 1 = r(uC_\varphi).$$

Theorem 4 gives us that the spectrum of uC_φ is the unit circle.

(2) Let us consider the space $H_p^\infty(\mathbb{D})$, which is non-Hilbert. Now $s = p$. Let $\varphi(z) = \frac{z+r}{1+rz}$ with $0 < r < 1$, so again φ is a hyperbolic automorphism of $\mathbb{D}$ with fixed points $a = 1$ and $b = -1$. Put $u(z) = (\varphi'(z))^p = (1 - r^2)^p/(1 + rz)^{2p}$. Then $u \in A(\mathbb{D})$ is bounded away from zero, and so uC_φ is invertible on $H_p^\infty(\mathbb{D})$. By Theorem 3, the spectral radius of uC_φ is

$$r(uC_\varphi) = \max\left\{\frac{|u(-1)|}{\varphi'(-1)^p}, \frac{|u(1)|}{\varphi'(1)^p}\right\} = 1$$

and

$$\sigma_{H_p^\infty}(uC_\varphi) = \{\lambda \in \mathbb{C}; |\lambda| = 1\}.$$

Finally we briefly discuss the proof of

$$r(uC_\varphi) = \max\left\{\frac{|u(a)|}{\varphi'(a)^s}, \frac{|u(b)|}{\varphi'(b)^s}\right\},$$

with hyperbolic automorphism $\varphi(z) = \frac{z+r}{1+rz}$, $0 < r < 1$.

One only need to show that $r(uC_\varphi) \le \max\left\{\frac{|u(a)|}{\varphi'(a)^s}, \frac{|u(b)|}{\varphi'(b)^s}\right\}$. We proceed as follows. For any $n \in \mathbb{N}$ we have

$$||(uC_\varphi)^n|| = ||\prod_{j=0}^{n-1} u \circ \varphi_j C_{\varphi_n}|| = ||\prod_{j=0}^{n-1} \frac{u \circ \varphi_j}{(\varphi' \circ \varphi_j)^s} \cdot \prod_{j=0}^{n-1} (\varphi' \circ \varphi_j)^s C_{\varphi_n}||.$$

Notice that $\prod_{j=0}^{n-1} \varphi' \circ \varphi_j = (\varphi_n)'$. Further, $w =: \frac{u}{(\varphi')^s}$ and observe that $w \in A(\mathbb{D})$ is also bounded away from zero. By condition $4)$, we obtain

$$||(uC_\varphi)^n||^{1/n} \le ||w_{(n)}||_\infty^{1/n} ||(\varphi_n')^s C_{\varphi_n}||^{1/n}.$$

Since it can be shown that

$$\lim_{n\to\infty} ||w_{(n)}||_\infty^{1/n} = \max\Big\{|w(a)|, |w(b)|\Big\} = \max\left\{\frac{|u(a)|}{\varphi'(a)^s}, \frac{|u(b)|}{\varphi'(b)^s}\right\}$$

and $\lim_{n\to\infty} ||(\varphi_n')^s C_{\varphi_n}||^{1/n} \le 1$, the claim follows.