Complex geometry of the symmetrised bidisc

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Extremality in Kobayashi’s hyperbolic complex spaces

In 1977 S. Kobayashi introduced the theory of hyperbolic complex spaces. In this context one studies the geometry and function theory of a domain $\Omega \subset \mathbb{C}^d$ with the aid of $2$-extremal holomorphic maps from the open unit disc $\mathbb{D}$ to $\Omega$. 
Extremality in Kobayashi’s hyperbolic complex spaces

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A prominent theme in hyperbolic complex geometry is a kind of duality between $\text{Hol}(\mathbb{D}, \Omega)$ and $\text{Hol}(\Omega, \mathbb{D})$, typified by the celebrated theorem of L. Lempert 1986, which in our terminology asserts that if $\Omega$ is convex then every 2-extremal map belonging to $\text{Hol}(\mathbb{D}, \Omega)$ is a complex geodesic of $\Omega$ (that is, has an analytic left inverse).

Here $\text{Hol}(\Omega, \mathbb{D})$ is the space of holomorphic maps from a domain $\Omega$ to $\mathbb{D}$.
$n$-extremal holomorphic maps

**Definition 1.** Let $\Omega$ be a domain, let $E \subset \mathbb{C}^N$, let $n \geq 1$, let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\Omega$ and let $z_1, \ldots, z_n \in E$. We say that the interpolation data

$$\lambda_j \mapsto z_j : \Omega \to E, \quad j = 1, \ldots, n,$$

are **extremally solvable** if there exists a map $h \in \text{Hol}(\Omega, E)$ such that $h(\lambda_j) = z_j$ for $j = 1, \ldots, n$, but, for any open neighbourhood $U$ of the closure of $\Omega$, there is no $f \in \text{Hol}(U, E)$ such that $f(\lambda_j) = z_j$ for $j = 1, \ldots, n$. 
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We say further that $h \in \text{Hol}(\Omega, E)$ is $n$-extremal (for $\text{Hol}(\Omega, E)$) if, for all choices of $n$ distinct points $\lambda_1, \ldots, \lambda_n$ in $\Omega$, the interpolation data

$$\lambda_j \mapsto h(\lambda_j) : \Omega \to E, \quad j = 1, \ldots, n,$$

are extremally solvable.

There are no 1-extremal holomorphic maps, so we shall always suppose that $n \geq 2$. 
For $\alpha \in \mathbb{D}$, the rational function

$$B_{\alpha}(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}$$

is called a Blaschke factor. A Möbius function is a function of the form $cB_{\alpha}$ for some $\alpha \in \mathbb{D}$ and $c \in \mathbb{T}$. The set of all Möbius functions is the automorphism group $\text{Aut} \mathbb{D}$ of $\mathbb{D}$.

We denote by $\mathcal{Bl}_n$ the set of Blaschke products of degree at most $n$. 
$n$-extremals for the Schur class and the Blaschke products

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In 1916 Pick showed that

a function $f$ is $n$-extremal for the Schur class $\mathcal{S} = \text{Hol}(\mathbb{D}, \Delta)$ if and only if $f \in \mathcal{B}l_{n-1}$.

Here $\Delta$ is the closed unit disc.
Symmetrised bidisc

In this talk we shall be mainly concerned with \( n \)-extremals for \( \text{Hol}(\mathbb{D}, \Gamma) \) where the symmetrised bidisc \( \mathcal{G} \) in \( \mathbb{C}^2 \) is defined to be the set

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\mathcal{G} \overset{\text{def}}{=} \{ (z + w, zw) : z, w \in \mathbb{D} \}
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Jim Agler and Nicholas Young began the study of the open symmetrised bidisc $ \mathcal{G} $ in 1995 with the aim of solving a special case of the $ \mu $-synthesis problem of $ H^\infty $ control.

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Agler and Young proved that the 2-extremals for \( \text{Hol}(\mathbb{D}, \mathcal{G}) \) coincide with the complex geodesics of \( \mathcal{G} \).
Interpolation in $\text{Hol}(\mathbb{D}, \Gamma)$

The (finite) interpolation problem for $\text{Hol}(\mathbb{D}, \Gamma)$ is the following: Given $\Gamma$-interpolation data

$$\lambda_j \mapsto z_j, \quad 1 \leq j \leq n,$$  \hspace{1cm} (1)

where $\lambda_1, \ldots, \lambda_n$ are $n$ distinct points in the open unit disc $\mathbb{D}$ and $z_1, \ldots, z_n$ are $n$ points in $\Gamma$, find if possible an analytic function

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If $\Gamma$ is replaced by the closed unit disc $\Delta$ then we obtain the classical Nevanlinna-Pick problem, for which there is an extensive theory that furnishes among many other things a simple criterion for the existence of a solution $h$ and an elegant parametrisation of all solutions when they exist.
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There is a satisfactory analytic theory of the problem (2) in the case that the number of interpolation points $n$ is 2, but we are still far from understanding the problem for a general $n \in \mathbb{N}$. 
**Condition $C_\nu$**

Here we introduce a sequence of necessary conditions for the solvability of an $n$-point $\Gamma$-interpolation problem and put forward a conjecture about sufficiency. We will show here that these conditions are of strictly increasing strength.

**Definition 2.** *Corresponding to $\Gamma$-interpolation data*

\[
\lambda_j \in \mathbb{D} \mapsto z_j = (s_j, p_j) \in \mathbb{G}, \quad 1 \leq j \leq n,
\]

we introduce:

**Condition $C_\nu(\lambda, z)$**

*For every Blaschke product $\nu$ of degree at most $\nu$, the Nevanlinna-Pick data*

\[
\lambda_j \mapsto \Phi(\nu(\lambda_j), z_j) = \frac{2\nu(\lambda_j)p_j - s_j}{2 - \nu(\lambda_j)s_j}, \quad j = 1, \ldots, n,
\]

*are solvable.*
Definition 3. The function $\Phi$ is defined for $(z, s, p) \in \mathbb{C}^3$ such that $zs \neq 2$ by

$$\Phi(z, s, p) = \frac{2zp - s}{2 - zs}.$$ 

We shall write $\Phi_z(s, p)$ as a synonym for $\Phi(z, s, p)$. 

The $\Gamma$-interpolation conjecture

**Conjecture 1.** Condition $C_{n-2}$ is necessary and sufficient for the solvability of an $n$-point $\Gamma$-interpolation problem.

Conjecture 1 is true in the case $n = 2$. We have no evidence for $n \geq 3$ and we are open minded as to whether or not it is likely to be true for all $n$. 
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Observe that Pick’s Theorem gives us an easily-checked criterion for the solvability of a Nevanlinna-Pick problem.

Proposition 1. If $\lambda_j \mapsto z_j = (s_j, p_j), \quad 1 \leq j \leq n$, are interpolation data for $\Gamma$ then condition $C_{\nu}(\lambda_1, \ldots, \lambda_n, z_1, \ldots, z_n)$ holds if and only if, for every Blaschke product $\nu$ of degree at most $\nu$,

$$
\left[ 1 - v(\lambda_i)p_i\bar{p}_j\bar{u}(\lambda_j) - \frac{1}{2}v(\lambda_i)(s_i - p_i\bar{s}_j) - \frac{1}{2}(\bar{s}_j - \bar{p}_j s_i)\bar{u}(\lambda_j) - \frac{1}{4}(1 - v(\lambda_i)\bar{v}(\lambda_j))s_i\bar{s}_j \right]^{n}_{i,j=1}
$$

is positive.
The conditions $C_\nu$ are all necessary for the solvability of a $\Gamma$-interpolation problem.

**Theorem 1.** Let $\lambda_1, \ldots, \lambda_n$ be distinct points in $\mathbb{D}$ and let $z_j \in \mathbb{G}$ for $j = 1, 2, \ldots, n$.

If there exists an analytic function $h : \mathbb{D} \to \Gamma$ such that $h(\lambda_j) = z_j$ for $j = 1, 2, \ldots, n$ then, for any function $\upsilon$ in the Schur class $S = \text{Hol}(\mathbb{D}, \Delta)$, the Nevanlinna-Pick data

$$
\lambda_j \mapsto \Phi(\upsilon(\lambda_j), z_j), \quad j = 1, \ldots, n, \quad (6)
$$

are solvable. In particular, the condition $C_\nu(\lambda, z)$ holds for every non-negative integer $\nu$. 

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Extremality in Condition $C_\nu$

To prove that condition $C_\nu$ suffices for the solvability of an $n$-point Nevanlinna-Pick problem for $\Gamma$ it is enough to prove it in the case that $C_\nu$ holds extremally. Let us make this notion precise.

Recall that $\Gamma$-interpolation data $\lambda_j \mapsto z_j$, $1 \leq j \leq n$, are defined to satisfy condition $C_\nu$ if, for every Blaschke product $\nu \in Bl_\nu$ of degree at most $\nu$, the data

$$\lambda_j \mapsto \Phi(\nu(\lambda_j), z_j), \quad 1 \leq j \leq n,$$

are solvable for the classical Nevanlinna-Pick problem. If, in addition, there exists $m \in Bl_\nu$ such that the data

$$\lambda_j \mapsto \Phi(m(\lambda_j), z_j), \quad 1 \leq j \leq n,$$

are extremally solvable Nevanlinna-Pick data, then we shall say that the data $\lambda_j \mapsto z_j$, $1 \leq j \leq n$, satisfy $C_\nu$ extremally, or the condition $C_\nu(\lambda, z)$ holds extremally.
It is well known that Pick’s criterion for the solvability of a classical Nevanlinna-Pick problem is expressible by an operator norm inequality; hence condition $C_\nu$ can be expressed this way. Let $H^2$ be the Hardy Hilbert space on $\mathbb{D}$ and let

$$ M = \text{span} \{ K_{\lambda_1}, \ldots, K_{\lambda_n} \} \subset H^2, \quad (8) $$

where $K_\lambda(z) = \frac{1}{1-\lambda z}$ ($\lambda, z \in \mathbb{D}$) is the Szegö kernel. Consider $\Gamma$-interpolation data

$$ \lambda_j \mapsto z_j, \ 1 \leq j \leq n, $$

and introduce, for any function $\nu$ in the Schur class, the operator $X(\nu)$ on $M$ given by

$$ X(\nu)K_{\lambda_j} = \Phi(\nu(\lambda_j), z_j)K_{\lambda_j}, \quad 1 \leq j \leq n. \quad (9) $$

Pick’s Theorem, as reformulated by Sarason, asserts that the Nevanlinna-Pick data

$$ \lambda_j \mapsto \Phi(\nu(\lambda_j), z_j), \quad 1 \leq j \leq n, \quad (10) $$

are solvable if and only if the operator $X(\nu)$ is a contraction. Furthermore, the Nevanlinna-Pick data (10) are extremally solvable if and only if $\|X(\nu)\| = 1.$
Thus $C_{\nu}(\lambda, z)$ holds if and only if

$$\sup_{v \in \mathcal{B}_{l_\nu}} \|X(v)\| \leq 1. \quad (11)$$

**Proposition 2.** For any $\Gamma$-interpolation data $\lambda_j \mapsto z_j$, $1 \leq j \leq n$, and $\nu \geq 0$, the following conditions are equivalent.

(i) $C_{\nu}(\lambda, z)$ holds extremally;

(ii) $\sup_{v \in \mathcal{B}_{l_\nu}} \|X(v)\| = 1$;

(iii) $C_{\nu}(\lambda, z)$ holds and there exist $m \in \mathcal{B}_{l_\nu}$ and $q \in \mathcal{B}_{l_{n-1}}$ such that

$$\Phi(m(\lambda_j), z_j) = q(\lambda_j), \quad j = 1, \ldots, n. \quad (12)$$

Moreover, when condition (iii) is satisfied for some $m \in \mathcal{B}_{l_\nu}$, there is a unique $q \in \mathcal{B}_{l_{n-1}}$ such that equations (12) hold. If, furthermore, the $\Gamma$-interpolation data $\lambda_j \mapsto z_j$, $1 \leq j \leq n$, are solvable by an analytic function $h = (s, p) : \mathbb{D} \to \Gamma$, then

$$\frac{2mp - s}{2 - ms} = q. \quad (13)$$
An auxiliary extremal for the condition $C_\nu(\lambda, z)$

We shall say that any Blaschke product $m$ with the properties described in Proposition 2(iii) is an auxiliary extremal for the condition $C_\nu(\lambda, z)$.

**Examples 2.** Let $\lambda_1, \lambda_2, \lambda_3$ be any three distinct points in $\mathbb{D}$ and let $0 < r < 1$. In each of the following examples $h$ is an analytic function from $\mathbb{D}$ to $G$ and the data $\lambda_j \mapsto h(\lambda_j), 1 \leq j \leq 3$, satisfy $C_1$ extremally.

(1) Let $h(\lambda) = (2r\lambda, \lambda^2)$. Every degree 0 inner function $m \in \mathbb{T}$ is an auxiliary extremal for $C_1$; there is no auxiliary extremal of degree 1.

(2) Let $h(\lambda) = (r(1 + \lambda), \lambda)$. Every $m \in Bl_1$ is an auxiliary extremal for $C_1$. The corresponding $q$ has degree $d(m) + 1$. 


An auxiliary extremal for the condition $C_\nu(\lambda, z)$

(3) Let

$$h(\lambda) = \left( 2(1 - r) \frac{\lambda^2}{1 + r\lambda^3}, \frac{\lambda(\lambda^3 + r)}{1 + r\lambda^3} \right), \quad \lambda \in \mathbb{D}.$$ 

The function $m(\lambda) = -\lambda$ is an auxiliary extremal for $C_1$; there is no auxiliary extremal of degree 0. Here $q(\lambda) = -\lambda^2$.

(4) Let $f$ be a Blaschke product of degree 1 or 2 and let $h = (2f, f^2)$. Every $m \in \mathcal{B}l_1$ is an auxiliary extremal and, for every $m$, we have $q = -f$. 
Γ-inner functions

Definition 4. A Γ-inner function is an analytic function $h : \mathbb{D} \to \Gamma$ such that the radial limit

$$
\lim_{r \to 1^-} h(r\lambda) \in b\Gamma
$$

(14)

for almost all $\lambda \in \mathbb{T}$.

Here $b\Gamma$ is the distinguished boundary of $\mathbb{G}$ (or $\Gamma$). It is the symmetrisation of the 2-torus:

$$
b\Gamma = \{(z + w, zw) : |z| = |w| = 1\}.
$$

By Fatou’s Theorem, the radial limit (14) exists for almost all $\lambda \in \mathbb{T}$ with respect to Lebesgue measure.

Observe that, if $h = (h_1, h_2)$ is a Γ-inner function, then $h_2$ is an inner function on $\mathbb{D}$ in the conventional sense.
The classes $\mathcal{E}_{\nu k}$

Proposition 2 tells us that if $h \in \text{Hol}(\mathbb{D}, \Gamma)$ and $\lambda_1, \ldots, \lambda_n$ are distinct points in $\mathbb{D}$, then the $\Gamma$-interpolation data $\lambda_j \mapsto h(\lambda_j)$ satisfy $C_{\nu}(\lambda, h(\lambda))$ extremally if and only if there exists $m \in \mathcal{B}l_{\nu}$ such that $\Phi \circ (m, h) \in \mathcal{B}l_{n-1}$. This leads us to introduce the following classes of rational $\Gamma$-inner functions.

**Definition 5.** For $\nu \geq 0$, $k \geq 1$ we say that the function $h$ is in $\mathcal{E}_{\nu k}$ if $h = (s, p) \in \text{Hol}(\mathbb{D}, \Gamma)$ is rational and there exists $m \in \mathcal{B}l_{\nu}$ such that

$$\frac{2mp - s}{2 - ms} \in \mathcal{B}l_{k-1}.$$ 

**Remark 3.** It is obvious that, for every $\nu \geq 0$,

$$\mathcal{E}_{\nu 1} \subset \mathcal{E}_{\nu 2} \subset \cdots \subset \mathcal{E}_{\nu k} \subset \mathcal{E}_{\nu, k+1} \subset \cdots,$$

and, for every $k \geq 1$,

$$\mathcal{E}_{0 k} \subset \mathcal{E}_{1k} \subset \cdots \subset \mathcal{E}_{\nu k} \subset \mathcal{E}_{\nu+1, k} \subset \cdots.$$
Superficial $\Gamma$-inner functions and the classes $\mathcal{E}_{\nu 1}$

For any inner function $\varphi$ and $\omega \in \mathbb{T}$ the function $h = (\omega + \varphi, \omega \varphi)$ is $\Gamma$-inner, and has the property that $h(\lambda)$ lies in the topological boundary $\partial \Gamma$ of $\Gamma$ for all $\lambda \in \mathbb{D}$.

Recall that $(s, p) \in \partial \Gamma \iff |s| \leq 2$ and $|s - \bar{s}p| = 1 - |p|^2$

$\iff$ there exist $z \in \mathbb{T}$ and $w \in \Delta$ such that $s = z + w$, $p = zw$.

**Definition 6.** A function $h \in \text{Hol}(\mathbb{D}, \Gamma)$ is **superficial** if $h(\mathbb{D}) \subset \partial \Gamma$.

The image of a function in $\text{Hol}(\mathbb{D}, \Gamma)$ is either contained in or disjoint from $\partial \Gamma$.

**Lemma 1.** If $h \in \text{Hol}(\mathbb{D}, \Gamma)$ is not superficial then $h(\mathbb{D}) \subset \mathbb{G}$.

**Proposition 3.** A $\Gamma$-inner function $h$ is superficial if and only if there is an $\omega \in \mathbb{T}$ and an inner function $p$ such that $h = (\omega p + \bar{\omega}, p)$.

**Theorem 4.** For every $\nu \geq 1$, the class $\mathcal{E}_{\nu 1}$ is equal to $\mathcal{E}_{01}$ and consists of the superficial rational $\Gamma$-inner functions.
The classes $\mathcal{E}_{\nu k}$ and $k$-extremals, $k \geq 2$

**Theorem 5.** If $h \in \mathcal{E}_{\nu k}$, where $\nu \geq 0$ and $k \geq 2$, and $h$ is not superficial then $h$ is $k$-extremal for $\text{Hol}(\mathbb{D}, \Gamma)$.

If Conjecture 1 is true then all $n$-extremals for $\Gamma$ lie in $\mathcal{E}_{n-2,n}$.

**Observation 6.** Let $n \geq 2$. If condition $C_{n-2}$ suffices for the solvability of $n$-point $\Gamma$-interpolation problems then every rational $\Gamma$-inner function $h$ which is $n$-extremal for $\text{Hol}(\mathbb{D}, \Gamma)$ belongs to $\mathcal{E}_{n-2,n}$. 
Complex geodesics of $\mathbb{G}$ and the classes $\mathcal{E}_{\nu 2}$

We recall that an analytic function $h : \mathbb{D} \to \Omega$ is called a complex geodesic of $\Omega$ if there exists an analytic left inverse $g : \Omega \to \mathbb{D}$ of $h$.

**Example 1.** Let $|\beta| < 1$. The function

$$h(\lambda) = (\beta \lambda + \bar{\beta}, \lambda)$$

is not only $\Gamma$-inner – it is a complex geodesic of $\mathbb{G}$. The simplest left inverse is the projection $(s, p) \mapsto p$. The domain $\mathbb{G}$ also has complex geodesics of degree 2.

**Proposition 4.** An analytic function $h : \mathbb{D} \to \mathbb{G}$ is a complex geodesic of $\mathbb{G}$ if and only if there is an $\omega \in \mathbb{T}$ such that $\Phi_\omega \circ h \in \text{Aut} \mathbb{D}$. Furthermore, every complex geodesic of $\mathbb{G}$ is $\Gamma$-inner.

**Theorem 7.** For $\nu \geq 0$ the set $\mathcal{E}_{\nu 2}$ is the union of the set of superficial rational $\Gamma$-inner functions and the set of complex geodesics of $\mathbb{G}$. 
**Condition $C_\nu$ and the classes $E_{\nu_k}$**

It is clear that $C_\nu(\lambda, z)$ implies $C_{\nu-1}(\lambda, z)$ for any $\Gamma$-interpolation data $\lambda \mapsto z$. To show that $C_\nu$ is strictly stronger than $C_{\nu-1}$ we need to find $\Gamma$-interpolation data

$$\lambda_j \in \mathbb{D} \mapsto z_j = (s_j, p_j) \in \mathbb{G}, \quad 1 \leq j \leq k,$$

such that

(i) for every Blaschke product $\nu$ of degree at most $\nu - 1$,

$$\lambda_j \mapsto \frac{2\nu(\lambda_j)p_j - s_j}{2 - \nu(\lambda_j)s_j}, \quad j = 1, \ldots, k,$$

are solvable Nevanlinna-Pick data, but

(ii) there is a Blaschke product $m$ of degree $\nu$ such that

$$\lambda_j \mapsto \frac{2m(\lambda_j)p_j - s_j}{2 - m(\lambda_j)s_j}, \quad j = 1, \ldots, k,$$

are not solvable Nevanlinna-Pick data.
Proposition 5. If there exists a nonconstant function $h \in \mathcal{E}_{\nu k} \setminus \mathcal{E}_{\nu -1,k}$ then $C_{\nu}$ is strictly stronger than $C_{\nu -1}$. In fact there is a set of $\Gamma$-interpolation data $\lambda_j \mapsto z_j$ with $k$ interpolation points which satisfies $C_{\nu -1}$ but not $C_{\nu}$.
Inequations for the classes $E_{\nu k}$

**Proposition 6.** For all $\nu \geq 1$ and $0 < r < 1$, the function

$$h_\nu(\lambda) = \left( \frac{2(1 - r)}{1 + r \lambda^{2\nu + 1}}, \frac{\lambda (\lambda^{2\nu + 1} + r)}{1 + r \lambda^{2\nu + 1}} \right), \quad \lambda \in \mathbb{D}, \quad (19)$$

belongs to $E_{\nu,\nu + 2} \setminus E_{\nu - 1,\nu + 2}$.

**Proof.** It is clear that $h_\nu$ is analytic on $\Delta$. Let $h_\nu = (s, p)$. It is simple to check that $s = \bar{s}p$ on $\mathbb{T}$, that $|s| \leq 2$ on $\mathbb{T}$ and that $|p(\lambda)| = 1$ on $\mathbb{T}$. This implies that $h_\nu(\mathbb{T}) \subset b\Gamma$ and that $h_\nu$ is $\Gamma$-inner.

Let $m(\lambda) = -\lambda^\nu$, so that $m \in B_{l_\nu}$. It is simple to verify that

$$\Phi \circ (m, h_\nu) = \frac{2mp - s}{2 - ms}(\lambda) = -\lambda^{\nu + 1} \in B_{l_{\nu + 1}},$$

and so $h_\nu \in E_{\nu,\nu + 2}$.
To prove that $h_\nu$ is not in $E_{\nu-1,\nu+2}$ we must show that, for all $\nu \in Bl_{\nu-1}$, the Blaschke product $\Phi \circ (\nu, h_\nu)$ has degree at least $\nu + 2$. We can do it using cancellations in the functions $\Phi \circ (\nu, h_\nu)$. It transpires that cancellations can only happen at special points on the unit circle: $\lambda^{2\nu+1} = -1$. 
Our main theorem follows easily.

**Theorem 8.** For all $\nu \geq 1$, the condition $C_\nu$ is strictly stronger than $C_{\nu-1}$. In fact there is a set of $\Gamma$-interpolation data $\lambda_j \mapsto z_j$ with $\nu + 2$ interpolation points which satisfies $C_{\nu-1}$ but not $C_{\nu}$.

As we observed above, $C_0$ is necessary and sufficient for solvability of a $\Gamma$-interpolation problem when $n = 2$, but a consequence of Theorem 8 is:

**Corollary 1.** For all $n \geq 3$, Condition $C_{n-3}$ does not suffice for the solvability of an $n$-point $\Gamma$-interpolation problem.
Table of relations between the classes $\mathcal{E}_{\nu k}$

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References


Thank you