Non-selfadjoint double commutant theorems

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This project is joint work with

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Definition. *Let* $\varnothing \neq \mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$,

$$\mathcal{S}' = \{T \in \mathcal{B}(\mathcal{H}) : TS = ST \text{ for all } S \in \mathcal{S}\}$$
$$\mathcal{S}^\perp = \{T \in \mathcal{B}(\mathcal{H}) : TS = ST = 0 \text{ for all } S \in \mathcal{S}\} \subseteq \mathcal{S}'.$$

von Neumann's Double Commutant Theorem:
If $I \in \mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a selfadjoint algebra, then

$$\mathcal{A}'' = \text{WOT-CL}\, \mathcal{A} = \text{SOT-CL}\, \mathcal{A}.$$

The motivation for this project is 2-fold:

(a) von Neumann's DCT is good. Very good.

(b) Let $\mathcal{A} = \begin{bmatrix} \alpha & 0 & * \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \subseteq \mathbb{M}_3$. Then $\mathcal{A} = \mathcal{A}''$, but $\mathcal{A} \neq \mathcal{A}^*$.

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Question: Which subalgebras $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ satisfy $\mathcal{A} = \mathcal{A}''$?

Known:

(a) If $\dim \mathcal{H} < \infty$, then $\mathcal{A}_T = \mathcal{A}_T''$ for all $T \in \mathcal{B}(\mathcal{H})$. (Wedderburn 1934)

(b) If $\dim \mathcal{H} = \infty$, then $\mathcal{A}_T = \mathcal{A}_T''$ if T is

- the unilateral shift (Brown-Halmos 1964)
- an algebraic operator (Ruston 1969)
- a unilateral weighted shift (Shields-Wallen 1970)
- a reductive normal operator (Turner 1972)
- others examples, but no complete characterization.

(c) Non-commutative analytic Toeplitz algebras (Davidson-Pitts / Popescu) $\mathfrak{L}_n$ - the left regular representation of $\mathcal{F}_n^+$ on $\ell_2(\mathcal{F}_n^+)$ via

$$\lambda(w)\xi_v = \xi_{wv}.$$

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Our algebras: (when $\dim \mathcal{H} < \infty$):

Fix a masa $\mathcal{D}_n$ a masa in $\mathcal{B}(\mathcal{H}) \simeq \mathbb{M}_n(\mathbb{C})$.

- $\mathcal{R}$ a $\mathcal{D}_n$ bimodule;
  $\mathcal{R} = \vee_\gamma \mathcal{R}_{P_\gamma, Q_\gamma}$ where $P_\gamma, Q_\gamma \in \mathcal{D}_n$ and
  $$\mathcal{R}_{P,Q} = Q\mathcal{B}(\mathcal{H})P.$$
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- Set
  $$\mathcal{S} = \mathcal{D} + \mathcal{R}.$$

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THE MYSTERIOUS CASE OF THE SCALAR DIAGONALS.

Example 1. $\mathcal{D} = \mathbb{C}I$, i.e. $\mathcal{S} = \mathcal{R} + \mathbb{C}I$.

$$\mathcal{R} = \begin{bmatrix} * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

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Hence $\mathcal{S}'' \neq \mathcal{S}$. [$\mathcal{S}$ does not have the DCP.]

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Yes, $\mathcal{S}$ has the DCP.

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Let us reconsider Example 3: $\mathcal{S} = \mathbb{C}I + \mathcal{R}$, where

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The incidence graph • corresponding to $\mathcal{R}$ is disconnected. If we write $\mathcal{R}$ with respect to the connected components of the graph, we get

$$\mathcal{R} \simeq \begin{bmatrix} 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathcal{R}_{P_{e_4}, Q_{e_1}} + \mathcal{R}_{P_{e_5}, Q_{e_2}}.$$

Connectedness is a good thing:

Theorem. *[M-Sourour 2011]*
Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a masa and let $\mathcal{R} = \vee\{\mathcal{R}_{P_\gamma, Q_\gamma} : \gamma \in \Gamma\}$ be a block-generated bimodule over $\mathcal{M}$. Then

(a) *the annihilator $\mathcal{R}^\perp$ of $\mathcal{R}$ satisfies*

$$\mathcal{R}^\perp = \mathcal{R}_{Q_0^\perp, P_0^\perp},$$

where $P_0 = \vee_\gamma P_\gamma$ and $Q_0 = \vee_\gamma Q_\gamma$.

(b) *$\mathcal{R}$ is connected if and only if $\mathcal{R}' = \mathcal{R}^\perp + \mathbb{C}I$.*

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It follows that if $\mathcal{R} = \vee\{\mathcal{R}_{P_\gamma, Q_\gamma} : \gamma \in \Gamma\}$ is connected and $\mathcal{S} = \mathbb{C}I + \mathcal{R}$, then

$$\begin{aligned} \mathcal{S}' = \mathcal{R}' &= \mathcal{R}^\perp + \mathbb{C}I \\ &= \mathcal{R}_{Q_0^\perp, P_0^\perp} + \mathbb{C}I, \end{aligned}$$

where $P_0 = \vee P_\gamma$, and $Q_0 = \vee Q_\gamma$.
But $\mathcal{R}_{Q_0^\perp, P_0^\perp}$ is connected, so

$$\begin{aligned} \mathcal{S}'' &= (\mathcal{R}_{Q_0^\perp, P_0^\perp})' \\ &= (\mathcal{R}_{Q_0^\perp, P_0^\perp})^\perp + \mathbb{C}I \\ &= \mathcal{R}_{P_0, Q_0} + \mathbb{C}I. \end{aligned}$$

In other words, if we want $\mathcal{S} = \mathcal{S}''$, then (when $\mathcal{R}$ is connected), we need $\mathcal{R}$ to be block-closed.

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Let us return to Example 4. $\mathcal{D} = \mathbb{C}I$, i.e. $\mathcal{S} = \mathcal{R} + \mathbb{C}I$.

$$\mathcal{R} = \begin{bmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\mathcal{S}'' = \begin{bmatrix} \alpha & 0 & 0 & * & * \\ 0 & \alpha & 0 & * & * \\ 0 & 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & 0 & \alpha \end{bmatrix}.$$

$P_0 = P_{e_4,e_5}$ and $Q_0 = Q_{e_1,e_2}$. The original $\mathcal{R}$ is connected, but it is not block-closed, and so $\mathcal{S} \neq \mathcal{S}''$.

Example 1: $\mathcal{R} = \begin{bmatrix} * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$,

and

Example 2: $\mathcal{R} = \begin{bmatrix} * & * & * & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

In both cases, $\mathcal{R}$ is a direct sum of two block-closed, connected components. Yet $\mathcal{S}$ does not have DCP in Example 1, while in Example 2, it does.

In each case, $\mathcal{R} = \mathcal{R}_{P_1,Q_1} \oplus \mathcal{R}_{P_2,Q_2}$, and hence

$$\begin{aligned} \mathcal{S}' &= (\mathcal{R} + \mathbb{C}I)' = \mathcal{R}' \\ &= (\mathcal{R}_{P_1,Q_1} + \mathcal{R}_{P_2,Q_2})' \\ &= \mathcal{R}_{P_1,Q_1}{}' \cap \mathcal{R}_{P_2,Q_2}{}' \\ &= (\mathcal{R}_{P_1,Q_1}{}^{\perp} + \mathbb{C}I) \cap (\mathcal{R}_{P_2,Q_2}{}^{\perp} + \mathbb{C}I) \\ &= (\mathcal{R}_{Q_1^{\perp},P_1^{\perp}} + \mathbb{C}I) \cap (\mathcal{R}_{Q_2^{\perp},P_2^{\perp}} + \mathbb{C}I) \end{aligned}$$

In Example One, however, $P_1 + P_2 = I$, from which it follows that

$$\mathcal{S}' = \mathbb{C}P_1 + \mathbb{C}P_1^{\perp}.$$

Thus $\mathcal{S}''$ is a block-diagonal von Neumann algebra.

In Example Two, $P_1 + P_2 \neq I$, and there remains enough in $\mathcal{S}'$ to ensure that $\mathcal{S}'' = \mathcal{S}$.

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$$\begin{aligned}
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&= {\mathcal{R}_{P_1,Q_1}}' \cap {\mathcal{R}_{P_2,Q_2}}' \\
&= ({\mathcal{R}_{P_1,Q_1}}^{\perp} + \mathbb{C}I) \cap ({\mathcal{R}_{P_2,Q_2}}^{\perp} + \mathbb{C}I) \\
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&= ({\mathcal{R}_{P_1,Q_1}}^{\perp} + \mathbb{C}I) \cap ({\mathcal{R}_{P_2,Q_2}}^{\perp} + \mathbb{C}I) \\
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In Example Two, $P_1 + P_2 \neq I$, and there remains enough in $\mathcal{S}'$ to ensure that $\mathcal{S}'' = \mathcal{S}$.

This also explains why in Example 3, if

$$\mathcal{R} = \begin{bmatrix} 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \simeq \begin{bmatrix} 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

if $\mathcal{S} = \mathbb{C}I + \mathcal{R}$, then $\mathcal{S}$ has the DCP.

Theorem. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a masa in $\mathcal{B}(\mathcal{H})$. Let $\mathcal{R}$ be a block-generated bimodule over $\mathcal{M}$ •, and let $\mathcal{S} = \mathbb{C}I + \mathcal{R}$. Then the following are equivalent.*

(a) *$\mathcal{S}$ satisfies the DCP;*

(b) *$\mathcal{S} = \mathbb{C}I + \mathcal{R} = \mathbb{C}I + \mathcal{R}_c$, and either*

  (i) *there exist mutually orthogonal projections $\{E_i\}_{i\in\mathbb{I}}$ such that $\sum_i E_i = I$ and*

$$\mathcal{R}_c = \oplus_{i\in\mathbb{I}}\mathcal{R}_{E_i,E_i}, \text{ or}$$

  (ii) *$\mathcal{R}_c = \oplus_{i\in\mathbb{I}}\mathcal{R}_{E_i,F_i}$ is block-closed and $\sum_i E_i \neq I \neq \sum_i F_i$.*

CONSTRUCTIONS THAT PRESERVE THE DCP

Proposition. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H}_1), \mathcal{B} \subseteq \mathcal{B}(\mathcal{H}_2)$ be unital algebras.*

(a) *If $\mathcal{A}$, $\mathcal{B}$ satisfy the DCP, then $\mathcal{A} \oplus \mathcal{B}$ satisfies the DCP.*

(b) *If $\mathcal{A}_1, \mathcal{A}_2 \subseteq \mathcal{B}(\mathcal{H}_1)$ satisfy the DCP, then so does $\mathcal{A}_1 \cap \mathcal{A}_2$.*

(c) *If $\dim \mathcal{H}_k < \infty$ and $\mathcal{A}$, $\mathcal{B}$ satisfy the DCP, then $\mathcal{A} \otimes \mathcal{B}$ satisfies the DCP.*

Proposition. *If $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ satisfies the DCP and $P \in \mathcal{A}$ is a projection, then*

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Proposition. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be an algebra that satisfies the DCP. Suppose that $\mathcal{M} \subseteq \mathcal{H}$ is an invariant subspace for $\mathcal{A}$, and let P denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. If*

$$\mathcal{B} := \{T \oplus (PTP)|_{\mathcal{M}} : T \in \mathcal{A}\},$$

then $\mathcal{B}$ satisfies the DCP as a subalgebra of $\mathcal{B}(\mathcal{H} \oplus \mathcal{M})$.

In other words, the map:

$$\begin{bmatrix} T_1 & T_2 \\ 0 & T_4 \end{bmatrix} \mapsto \begin{bmatrix} T_1 & T_2 & 0 \\ 0 & T_4 & 0 \\ 0 & 0 & T_1 \end{bmatrix}$$

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Skeletons.

Let $\mathcal{A} \subseteq \mathbb{M}_n(\mathbb{C})$ be an algebra. Then $\mathcal{A} = \mathcal{M} + \mathcal{N}$, where

$$\mathcal{M} \simeq \mathbb{M}_{n_1} \otimes I_{m_1} \oplus \ldots \oplus \mathbb{M}_{n_r} \otimes I_{m_r}$$

is the semisimple part of $\mathcal{A}$, and $\mathcal{N}$ is the Jacobson radical of $\mathcal{A}$. Suppose that we now pick idempotents of rank one: $p_1 \in \mathbb{M}_{n_1}, p_2 \in \mathbb{M}_{n_2}, \ldots, p_r \in \mathbb{M}_{n_r}$ and let $p = p_1 \otimes I_{m_1} + \ldots + p_r \otimes I_{m_r}$.

Definition. *We define the* **skeleton** *of $\mathcal{A}$ to be*

$$\mathrm{sk}(\mathcal{A}) = \mathrm{sk}_p(\mathcal{A}) = p\mathcal{A}p|_{\mathrm{ran}\, p} \subseteq \mathbb{M}_{m_1+\ldots+m_r}.$$

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Example: if

$$\mathcal{A} \sim \begin{bmatrix} a & b & x_1 & x_2 & y_1 & y_2 & y_3 \\ c & d & x_3 & x_4 & y_4 & y_5 & y_6 \\ 0 & 0 & a & b & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_1 & z_2 & z_3 \\ 0 & 0 & 0 & 0 & z_4 & z_5 & z_6 \\ 0 & 0 & 0 & 0 & z_7 & z_8 & z_9 \end{bmatrix},$$

then

$$\mathrm{sk}(\mathcal{A}) \sim \begin{bmatrix} a & x_1 & y_1 \\ 0 & a & 0 \\ 0 & 0 & z_1 \end{bmatrix}.$$

Up to algebra isomorphism, the skeleton of $\mathcal{A}$ is well-defined.

Theorem. *Let $\mathcal{A} \subseteq \mathbb{M}_n(\mathbb{C})$ be an algebra and $\mathrm{sk}(\mathcal{A})$ denote the skeleton of $\mathcal{A}$. Then $\mathcal{A}$ satisfies the DCP if and only if $\mathrm{sk}(\mathcal{A})$ satisfies the DCP.*

Theorem. *Let $\mathcal{D} \subseteq \mathcal{D}_n$ be a unital algebra and $\mathcal{R} \subseteq \mathbb{M}_n(\mathbb{C})$ be a block-closed $\mathcal{D}_n$-bimodule. Let $\mathcal{S} = \mathcal{D} + \mathcal{R}$ and assume that $\mathcal{S} = \mathrm{sk}(\mathcal{S})$. Let $\mathcal{S} = \mathcal{D}_\mathcal{S} + \mathcal{R}_\mathcal{S}$ be the standard form of $\mathcal{S}$, and write $\mathcal{D}_\mathcal{S} = \mathrm{span}\{D_1, D_2, ..., D_s\}$, where the D_i's are minimal projections in $\mathcal{D}_\mathcal{S}$ and $\mathcal{R}_\mathcal{S} = \oplus_{i=1}^r \mathcal{R}_{E_i,F_i}$ be the decomposition of $\mathcal{R}_\mathcal{S}$ into connected components. The following are equivalent:*

(a) *$\mathcal{S}$ satisfies the DCP, i.e. $\mathcal{S} = \mathcal{S}''$.*

(b) *For each $1 \leq j \leq s$, $D_j \not\leq \sum_{i=1}^r E_i$ and $D_j \not\leq \sum_{i=1}^r F_i$.*

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Wedderburn's Theorem revisited. Let $T \in \mathbb{M}_n(\mathbb{C})$ and suppose $\sigma(T) = \{\alpha_1, ..., \alpha_r\}$. Then $T \sim \oplus_{k=1}^r T_k$, where $\sigma(T_k) = \{\alpha_k\}$, and

$$\mathcal{A}_T \sim \oplus_k \mathcal{A}_{T_k}.$$

It suffices to show that each $\mathcal{A}_{T_k}$ has the DCP, i.e. that $\mathcal{A}_T$ has the DCP when $\sigma(T) = \{\alpha\}$. WLOG, $\alpha = 0$.

Now $T \simeq \oplus_{k=1}^s (J_{m_k})$, where J_r is the $r \times r$ Jordan cell. WLOG, $m_1 \geq m_2 \geq \cdots \geq m_s$.
Note that for any $m > 0$, $\mathcal{A}_{J_m}$ has the DCP, because it is a maximal abelian subalgebra of $\mathbb{M}_m(\mathbb{C})$.
Finally,

$$\mathcal{A}_T = \{R \oplus (P_2 R P_2)|_{\operatorname{ran} P_2} \oplus \cdots \oplus (P_s R P_s)|_{\operatorname{ran} P_s} : R \in \mathbf{Tp}_{m_1}\}.$$

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SUMMARY For algebras which are equal to their skeleta:
We know of two classes of algebras which satisfy the DCP.

- Those of our main theorem in finite-dimensions.
- Maximal abelian subalgebras, such as the Toeplitz algebra T_p.

We know that we can obtain more examples through

- compressions by a projection in the algebra,
- tensor products,
- intersections,
- direct sums,
- pinchings, and
- going to the skeleton of an algebra.

Is every DCP algebra in $\mathbb{M}_n(\mathbb{C})$ obtained in this way from the basic examples above?

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They laughed when I said I was going to be a comedian. Well, they're not laughing now.

Bob Monkhouse

Tack för er uppmärksamhet!