Ruston elements and Fredholm theory relative to arbitrary homomorphisms

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Outline of Talk

- Preliminaries and References

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- Background on Fredholm theory

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- Ruston and almost Ruston elements and spectra

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- Mapping properties of the Ruston and almost Ruston spectra

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- Ruston and almost Ruston elements and spectra
- Mapping properties of the Ruston and almost Ruston spectra
- Applications of the theory of Ruston and almost Ruston elements

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∂K: the topological boundary of a set $K \subseteq \mathbb{C}$

ηK: the connected hull of a compact set $K \subseteq \mathbb{C}$ (i.e. K together with its "holes")

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If $T : A \to B$ is bounded, then $a \in A$ is a *Riesz element* if $\sigma(a + T^{-1}(0), A/T^{-1}(0)) = \{0\}$.

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$\mathcal{R}_T$: the set of all Riesz elements

Some References

- H. du T. Mouton, S. Mouton and H. Raubenheimer: Ruston elements and Fredholm theory relative to arbitrary homomorphisms, *Quaestiones Math.* **34** (2011), 1–19.

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- R. E. Harte: Fredholm theory relative to a Banach algebra homomorphism, *Math. Z.* **179** (1982), 431–436.

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- R. E. Harte: Fredholm theory relative to a Banach algebra homomorphism, *Math. Z.* **179** (1982), 431–436.
- T. Mouton and H. Raubenheimer: More on Fredholm theory relative to a Banach algebra homomorphism, *Proc. R. Ir. Acad.* **93A** (1993), 17–25.

Let A and B be Banach algebras and $T : A \to B$ a homomorphism.

Fredholm theory

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Definition (Harte, 1982)

With respect to $T : A \to B$, we shall describe an element $a \in A$ as:

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3. Browder if $a \in \{b + c : b \in A^{-1}, c \in T^{-1}(0), bc = cb\}$.

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We have the following, relative to $T : A \to B$, for $a \in A$:
invertible $\Rightarrow$ Browder $\Rightarrow$ Weyl $\Rightarrow$ Fredholm

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Theorem (Harte, 1982, and T. Mouton & Raubenheimer, 1993)

1. *If a is almost invertible Fredholm, then a is Browder. Hence we have, relative to T, for $a \in A$:*
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   invertible $\Rightarrow$ almost invertible Fredholm $\Rightarrow$ Browder $\Rightarrow$ Weyl $\Rightarrow$ Fredholm
2. *Suppose that T satisfies the Riesz property and $a \in A$. Then a is almost invertible Fredholm if and only if a is Browder.*

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Let $T : A \to B$ and $a \in A$.

1. The Fredholm spectrum of a is the spectrum $\sigma(Ta, B)$.

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$$\omega_T(a, A) := \{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not Weyl}\}.$$

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$$\omega_T(a, A) := \{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not Weyl}\}.$$
3. The Browder spectrum of a is the set
$$\omega_T^{\text{comm}}(a, A) := \{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not Browder}\}.$$

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$$\sigma(Ta, B) \subseteq \omega_T(a, A) \subseteq \omega_T^{\text{comm}}(a, A) \subseteq \sigma(a, A)$$

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2. *Suppose that T satisfies the Riesz property and $a \in A$. Then a is almost invertible Fredholm if and only if a is Browder.*

Corollary

1. $$\sigma(Ta, B) \subseteq \omega_T(a, A) \subseteq \omega_T^{comm}(a, A) \subseteq \sigma(Ta, B) \cup acc\, \sigma(a) \subseteq \sigma(a, A)$$
2. *If T satisfies the Riesz property, then*
   $$\omega_T^{comm}(a, A) = \sigma(Ta, B) \cup acc\, \sigma(a).$$

Interesting questions

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Known results:

Theorem (Harte, 1976)

If $T : A \to B$ is bounded with closed range and $a \in A$, then

$$\omega_T(a, A) \subseteq \eta\sigma(Ta, B).$$

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Suppose that $T : A \to B$ is bounded, has closed range and satisfies the Riesz property. If $a \in A$, then

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Suppose that $T : A \to B$ is bounded, has closed range and satisfies the Riesz property. If $a \in A$, then

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What if T is not bounded?

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Theorem (Harte, 1991)

If $T : A \to B$ is bounded with closed range, then T has the Riesz property if and only if T has the strong Riesz property.

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Ruston and almost Ruston elements

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Definition (T. Mouton & Raubenheimer, 1993)

Let $T : A \to B$ be bounded. With respect to T, we shall describe an element $a \in A$ as:

1. Ruston if $a \in \{b + c : b \in A^{-1}, c \in \mathcal{R}_T \text{ and } bc = cb\}$;

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2. Almost Ruston if $a \in \{b + c : b \in A^{-1}, c \in \mathcal{R}_T \text{ and } TbTc = TcTb\}$.

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Definition

With respect to $T : A \to B$, we shall describe an element $a \in A$ as:

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Now:

				Ruston				
			$\nearrow$		$\searrow$			
inv	$\Rightarrow$	Browder				alm Ruston	$\Rightarrow$	Fredholm
			$\searrow$		$\nearrow$			
				Weyl				

Ruston and almost Ruston elements

Definition

Let $T : A \to B$ and $a \in A$.

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$$\vartheta_T(a, A) := \{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not almost Ruston}\}.$$

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2. The Ruston spectrum of a is the set

$$\vartheta_T^{\text{comm}}(a, A) := \{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not Ruston}\}.$$

Recall:

$$\sigma(Ta, B) \subseteq \omega_T(a, A) \subseteq \omega_T^{\text{comm}}(a, A) \subseteq \sigma(a, A)$$

Ruston and almost Ruston elements

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Now:

$$\begin{array}{ccccccccc} & & & & \vartheta_T^{\text{comm}}(a) & & & & \\ & & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\subseteq$} & & & \\ \sigma(Ta) & \subseteq & \vartheta_T(a) & & & & \omega_T^{\text{comm}}(a) & \subseteq & \sigma(a) \\ & & & \rotatebox{-45}{$\subseteq$} & & \rotatebox{45}{$\subseteq$} & & & \\ & & & & \omega_T(a) & & & & \end{array}$$

Ruston and almost Ruston elements

Theorem

If T is bounded, has closed range and satisfies the Riesz property, then the sets of Ruston (respectively almost Ruston) elements relative to T as a bounded homomorphism and relative to T as a homomorphism with closed range coincide.

Ruston and almost Ruston elements

Theorem

If $T : A \to B$ has closed range and satisfies the Riesz property, then

1. *every Ruston element is Browder;*
2. *every almost Ruston element is Weyl.*

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If $T : A \to B$ has closed range and satisfies the Riesz property, then

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Corollary

Suppose that $T : A \to B$ has closed range and satisfies the Riesz property. If $a \in A$, then the Browder and Ruston spectra of a relative to T coincide and so do the Weyl and almost Ruston spectra.

Ruston and almost Ruston elements

Hence, if $T : A \to B$ has closed range and satisfies the Riesz property, then we have for $a \in A$:

$$
\begin{array}{ccccccccc}
 & & & & \text{Ruston} & & & & \\
 & & & \nearrow\!\!\!\!\!\swarrow & & \searrow & & & \\
\text{inv} & \Rightarrow & \text{Browder} & & & & \text{alm Ruston} & \Rightarrow & \text{Fredholm} \\
 & & & \searrow & & \nearrow\!\!\!\!\!\swarrow & & & \\
 & & & & \text{Weyl} & & & &
\end{array}
$$

Ruston and almost Ruston elements

Also, if $T : A \to B$ has closed range and satisfies the Riesz property, then we have for $a \in A$:

$$\begin{array}{ccccccccc}
 & & & & \vartheta_T^{\mathrm{comm}}(a) & & & & \\
 & & & \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$=$} & & & \\
\sigma(Ta) & \subseteq & \vartheta_T(a) & & & & \omega_T^{\mathrm{comm}}(a) & \subseteq & \sigma(a) \\
 & & & \rotatebox{-45}{$=$} & & \rotatebox{45}{$\subseteq$} & & & \\
 & & & & \omega_T(a) & & & &
\end{array}$$

Mapping Properties

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Theorem

If either B is commutative or T has closed range and satisfies the Riesz property, then

$$\vartheta_T(f(a), A) \subseteq f(\vartheta_T(a, A))$$

for all $a \in A$ and every function f analytic on a neighbourhood of $\sigma(a, A)$ which is non-constant on each component of its domain of definition.

Mapping Properties

Theorem

If $a \in A$, then

$$\vartheta_T^{comm}(f(a), A) = f(\vartheta_T^{comm}(a, A))$$

for all f analytic and one-to-one on a neighbourhood of $\sigma(a, A)$.

Mapping Properties

Theorem

Suppose that T has closed range and satisfies the Riesz property. Then

$$\vartheta_T^{comm}(f(a), A) = f(\vartheta_T^{comm}(a, A))$$

for all $a \in A$ and every function f analytic on a neighbourhood of $\sigma(a, A)$ which is non-constant on each component of its domain of definition.

Theorem

If B is commutative, then

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Applications

Recall:

Theorem (Harte, 1976)

If $T : A \to B$ is bounded with closed range and $a \in A$, then

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Now:

Theorem (T. Mouton, S. Mouton, Raubenheimer, 2011)

Suppose that $T : A \to B$ has closed range and satisfies the Riesz property. If $a \in A$, then

$$\omega_T(a, A) \subseteq \eta\sigma(Ta, B).$$

The proof uses Ruston theory, as well as:

Theorem (T. Mouton, S. Mouton, Raubenheimer, 2011)

If $a \in A$ *and* $e(a, A) := \{\lambda \in \mathbb{C} : \lambda 1 - a \notin \exp(A)\}$, *where* $\exp(A) = \{e^c : c \in A\}$, *then* $e(a, A)$ *is compact and*

$$e(a, A) \subseteq \eta\sigma(a, A).$$

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$$e(a, A) \subseteq \eta\sigma(a, A).$$

Theorem (T. Mouton, S. Mouton, Raubenheimer, 2011)

If $T : A \to B$ *and* $a \in A$, U *is a neighbourhood of* $\sigma(a, A)$ *and* $f : U \to \mathbb{C}$ *is analytic, then*

$$q := Tf(a) - f(Ta) \in QN(B)$$

with $q(Tf(a)) = (Tf(a))q$.

Applications

Recall:

Theorem (T. Mouton & Raubenheimer, 1993)

Suppose that $T : A \to B$ is bounded, has closed range and satisfies the Riesz property. If $a \in A$, then

$$\omega_T^{comm}(a, A) \subseteq \eta\sigma(Ta, B).$$

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Recall:

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The proof uses Ruston theory, as well as:

Theorem (T. Mouton, S. Mouton, Raubenheimer, 2011)

If $T : A \to B$ satisfies the Riesz property and $a \in A$, then

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The proof uses Ruston theory, as well as:

Theorem (T. Mouton, S. Mouton, Raubenheimer, 2011)

If $T : A \to B$ satisfies the Riesz property and $a \in A$, then

$$\omega_T^{comm}(a, A) \subseteq \eta\omega_T(a, A).$$

Recall: $\sigma(Ta, B) \subseteq \omega_T(a, A) \subseteq \omega_T^{\text{comm}}(a, A) \subseteq \sigma(a, A)$

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If $T : A \to B$ satisfies the Riesz property and $a \in A$, then

$$\omega_T^{comm}(a, A) \subseteq \eta\omega_T(a, A).$$

Recall: $\sigma(Ta, B) \subseteq \omega_T(a, A) \subseteq \omega_T^{\mathrm{comm}}(a, A) \subseteq \sigma(a, A)$

Theorem (T. Mouton, S. Mouton, Raubenheimer, 2011)

Suppose that $T : A \to B$ satisfies the Riesz property. Let $J = \overline{T^{-1}(0)}$ and let $\pi : A \to A/J$ be the canonical homomorphism. Then

1. $\omega_T^{comm}(a, A) = \omega_\pi^{comm}(a, A)$;
2. $\omega_T(a, A) = \omega_\pi(a, A)$.

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Recall:

Theorem (Harte, 1991)

If $T : A \to B$ is bounded with closed range, then T has the Riesz property if and only if T has the strong Riesz property.

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Now:

Theorem (T. Mouton, S. Mouton, Raubenheimer, 2011)

If $T : A \to B$ has closed range, then T has the Riesz property if and only if T has the strong Riesz property.

Hence, if $T : A \to B$ has closed range and satisfies the Riesz property, then T has the strong Riesz property.

Applications

If $T : A \to B$ satisfies the strong Riesz property, then we have for $a \in A$:

$$
\begin{array}{ccccccccc}
 & & & & \text{Ruston} & & & & \\
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S. Č. Živković-Zlatanović, D. S. Djordjević, R. E. Harte: Ruston, Riesz and perturbation classes, *J. Math. Anal. Appl.* **389** (2012), 871–886.

Applications

Theorem

If $T : A \to B$ satisfies the strong Riesz property and $a \in A$, then

$$\eta\sigma(Ta, B) = \eta\omega_T(a, A) = \eta\omega_T^{comm}(a, A).$$

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Theorem

If $T : A \to B$ satisfies the strong Riesz property and $a \in A$, then

$$\eta\sigma(Ta, B) = \eta\omega_T(a, A) = \eta\omega_T^{comm}(a, A).$$

S. Č. Živković-Zlatanović, R. E. Harte: Polynomially Riesz elements, submitted.

THANK YOU