

Ruston elements and Fredholm theory relative to arbitrary homomorphisms

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- Preliminaries and References

Outline of Talk

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- Mapping properties of the Ruston and almost Ruston spectra
- Applications of the theory of Ruston and almost Ruston elements

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ηK : the connected hull of a compact set $K \subseteq \mathbb{C}$ (i.e. K together with its “holes”)

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If $T : A \rightarrow B$ is bounded, then $a \in A$ is a *Riesz element* if $\sigma(a + T^{-1}(0), A/T^{-1}(0)) = \{0\}$.

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\mathcal{R}_T : the set of all Riesz elements

- H. du T. Mouton, S. Mouton and H. Raubenheimer: Ruston elements and Fredholm theory relative to arbitrary homomorphisms, *Quaestiones Math.* **34** (2011), 1–19.

Some References

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- R. E. Harte: Fredholm theory relative to a Banach algebra homomorphism, *Math. Z.* **179** (1982), 431–436.
- T. Mouton and H. Raubenheimer: More on Fredholm theory relative to a Banach algebra homomorphism, *Proc. R. Ir. Acad.* **93A** (1993), 17–25.

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Definition (Harte, 1982)

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- 3 Browder if $a \in \{b + c : b \in A^{-1}, c \in T^{-1}(0), bc = cb\}$.

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Theorem (Harte, 1982, and T. Mouton & Raubenheimer, 1993)

- 1 *If a is almost invertible Fredholm, then a is Browder. Hence we have, relative to T , for $a \in A$:
invertible \Rightarrow almost invertible Fredholm \Rightarrow Browder \Rightarrow Weyl \Rightarrow Fredholm*

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invertible \Rightarrow almost invertible Fredholm \Rightarrow Browder \Rightarrow Weyl \Rightarrow Fredholm*
- 2 *Suppose that T satisfies the Riesz property and $a \in A$. Then a is almost invertible Fredholm if and only if a is Browder.*

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- 1 The Fredholm spectrum of a is the spectrum $\sigma(Ta, B)$.

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Hence:

$$\sigma(Ta, B) \subseteq \omega_T(a, A) \subseteq \omega_T^{\text{comm}}(a, A) \subseteq \sigma(a, A)$$

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Corollary

- 1
$$\begin{aligned}\sigma(Ta, B) &\subseteq \omega_T(a, A) \subseteq \omega_T^{\text{comm}}(a, A) \\ &\subseteq \sigma(Ta, B) \cup \text{acc } \sigma(a) \subseteq \sigma(a, A)\end{aligned}$$
- 2 If T satisfies the Riesz property, then
$$\omega_T^{\text{comm}}(a, A) = \sigma(Ta, B) \cup \text{acc } \sigma(a).$$

Interesting questions

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$$\sigma(Ta, B) \subseteq \omega_T(a, A) \subseteq \omega_T^{\text{comm}}(a, A) \subseteq \sigma(a, A)$$

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Theorem (Harte, 1976)

If $T : A \rightarrow B$ is bounded with closed range and $a \in A$, then
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Suppose that $T : A \rightarrow B$ is bounded, has closed range and satisfies the Riesz property. If $a \in A$, then
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What if T is not bounded?

Known results:

Theorem (Harte, 1991)

If $T : A \rightarrow B$ is bounded with closed range, then T has the Riesz property if and only if T has the strong Riesz property.

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Ruston and almost Ruston elements

Definition (T. Mouton & Raubenheimer, 1993)

Let $T : A \rightarrow B$ be bounded. With respect to T , we shall describe an element $a \in A$ as:

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- 2 Almost Ruston if $a \in \{b + c : b \in A^{-1}, c \in \mathcal{R}_T \text{ and } TbTc = TcTb\}$.

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With respect to $T : A \rightarrow B$, we shall describe an element $a \in A$ as:

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Ruston and almost Ruston elements

Recall:

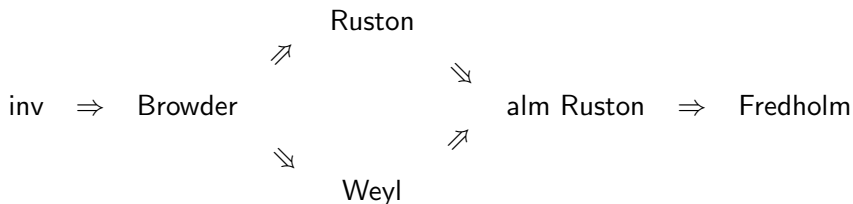
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Definition

Let $T : A \rightarrow B$ and $a \in A$.

- 1 The almost Ruston spectrum of a is the set

$$\vartheta_T(a, A) := \{\lambda \in \mathbb{C} : \lambda 1 - a \text{ is not almost Ruston}\}.$$

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Recall:

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Now:

$$\begin{array}{ccc} & \vartheta_T^{\text{comm}}(a) & \\ \subsetneq & & \supsetneq \\ \sigma(Ta) \subseteq \vartheta_T(a) & & \omega_T^{\text{comm}}(a) \subseteq \sigma(a) \\ \supsetneq & & \subsetneq \\ & \omega_T(a) & \end{array}$$

Theorem

If T is bounded, has closed range and satisfies the Riesz property, then the sets of Ruston (respectively almost Ruston) elements relative to T as a bounded homomorphism and relative to T as a homomorphism with closed range coincide.

Theorem

If $T : A \rightarrow B$ has closed range and satisfies the Riesz property, then

- 1 *every Ruston element is Browder;*
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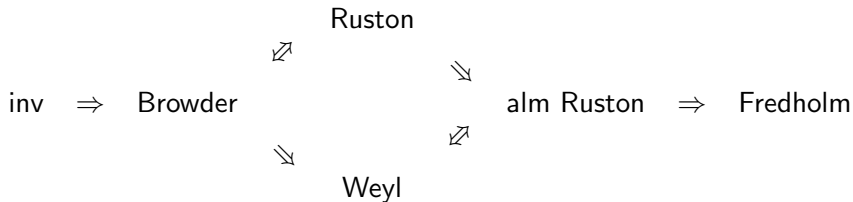
- 1 every Ruston element is Browder;
- 2 every almost Ruston element is Weyl.

Corollary

Suppose that $T : A \rightarrow B$ has closed range and satisfies the Riesz property. If $a \in A$, then the Browder and Ruston spectra of a relative to T coincide and so do the Weyl and almost Ruston spectra.

Ruston and almost Ruston elements

Hence, if $T : A \rightarrow B$ has closed range and satisfies the Riesz property, then we have for $a \in A$:



Ruston and almost Ruston elements

Also, if $T : A \rightarrow B$ has closed range and satisfies the Riesz property, then we have for $a \in A$:

$$\sigma(Ta) \subseteq \vartheta_T(a) \begin{array}{c} \subsetneq \\ \parallel \end{array} \begin{array}{c} \vartheta_T^{\text{comm}}(a) \\ \parallel \\ \omega_T(a) \end{array} \begin{array}{c} \parallel \\ \subsetneq \end{array} \omega_T^{\text{comm}}(a) \subseteq \sigma(a)$$

Mapping Properties

Theorem

If either B is commutative or T has closed range and satisfies the Riesz property, then

$$\vartheta_T(f(a), A) \subseteq f(\vartheta_T(a, A))$$

for all $a \in A$ and every function f analytic on a neighbourhood of $\sigma(a, A)$ which is non-constant on each component of its domain of definition.

Mapping Properties

Theorem

If $a \in A$, then

$$\vartheta_T^{\text{comm}}(f(a), A) = f(\vartheta_T^{\text{comm}}(a, A))$$

for all f analytic and one-to-one on a neighbourhood of $\sigma(a, A)$.

Theorem

Suppose that T has closed range and satisfies the Riesz property. Then

$$\vartheta_T^{\text{comm}}(f(a), A) = f(\vartheta_T^{\text{comm}}(a, A))$$

for all $a \in A$ and every function f analytic on a neighbourhood of $\sigma(a, A)$ which is non-constant on each component of its domain of definition.

Theorem

If B is commutative, then

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If $T : A \rightarrow B$ is bounded with closed range and $a \in A$, then

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Now:

Theorem (T. Mouton, S. Mouton, Raubenheimer, 2011)

Suppose that $T : A \rightarrow B$ has closed range and satisfies the Riesz property. If $a \in A$, then

$$\omega_T(a, A) \subseteq \eta\sigma(Ta, B).$$

The proof uses Ruston theory, as well as:

Theorem (T. Mouton, S. Mouton, Raubenheimer, 2011)

If $a \in A$ and $e(a, A) := \{\lambda \in \mathbb{C} : \lambda 1 - a \notin \exp(A)\}$, where $\exp(A) = \{e^c : c \in A\}$, then $e(a, A)$ is compact and

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$$e(a, A) \subseteq \eta\sigma(a, A).$$

Theorem (T. Mouton, S. Mouton, Raubenheimer, 2011)

If $T : A \rightarrow B$ and $a \in A$, U is a neighbourhood of $\sigma(a, A)$ and $f : U \rightarrow \mathbb{C}$ is analytic, then

$$q := Tf(a) - f(Ta) \in QN(B)$$

with $q(Tf(a)) = (Tf(a))q$.

Recall:

Theorem (T. Mouton & Raubenheimer, 1993)

Suppose that $T : A \rightarrow B$ is bounded, has closed range and satisfies the Riesz property. If $a \in A$, then

$$\omega_T^{\text{comm}}(a, A) \subseteq \eta\sigma(Ta, B).$$

Recall:

Theorem (T. Mouton & Raubenheimer, 1993)

Suppose that $T : A \rightarrow B$ is bounded, has closed range and satisfies the Riesz property. If $a \in A$, then

$$\omega_T^{comm}(a, A) \subseteq \eta\sigma(Ta, B).$$

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Suppose that $T : A \rightarrow B$ has closed range and satisfies the Riesz property. If $a \in A$, then

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Applications

The proof uses Ruston theory, as well as:

Theorem (T. Mouton, S. Mouton, Raubenheimer, 2011)

If $T : A \rightarrow B$ satisfies the Riesz property and $a \in A$, then

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The proof uses Ruston theory, as well as:

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If $T : A \rightarrow B$ satisfies the Riesz property and $a \in A$, then

$$\omega_T^{\text{comm}}(a, A) \subseteq \eta\omega_T(a, A).$$

Recall: $\sigma(Ta, B) \subseteq \omega_T(a, A) \subseteq \omega_T^{\text{comm}}(a, A) \subseteq \sigma(a, A)$

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Recall: $\sigma(Ta, B) \subseteq \omega_T(a, A) \subseteq \omega_T^{\text{comm}}(a, A) \subseteq \sigma(a, A)$

Theorem (T. Mouton, S. Mouton, Raubenheimer, 2011)

Suppose that $T : A \rightarrow B$ satisfies the Riesz property. Let $J = \overline{T^{-1}(0)}$ and let $\pi : A \rightarrow A/J$ be the canonical homomorphism. Then

- 1 $\omega_T^{\text{comm}}(a, A) = \omega_\pi^{\text{comm}}(a, A)$;
- 2 $\omega_T(a, A) = \omega_\pi(a, A)$.

Recall:

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If $T : A \rightarrow B$ is bounded with closed range, then T has the Riesz property if and only if T has the strong Riesz property.

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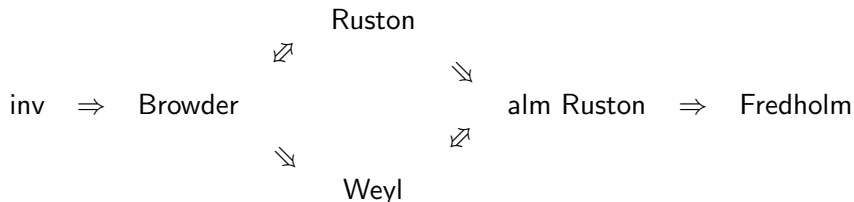
Theorem (T. Mouton, S. Mouton, Raubenheimer, 2011)

If $T : A \rightarrow B$ has closed range, then T has the Riesz property if and only if T has the strong Riesz property.

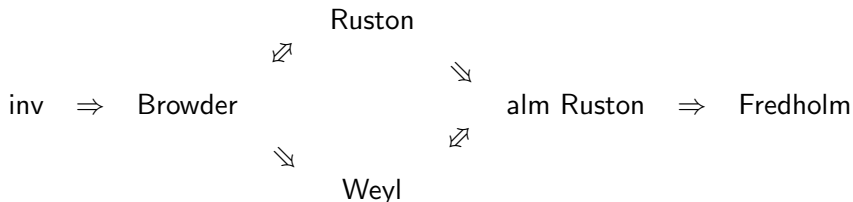
Hence, if $T : A \rightarrow B$ has closed range and satisfies the Riesz property, then T has the strong Riesz property.

Applications

If $T : A \rightarrow B$ satisfies the strong Riesz property, then we have for $a \in A$:



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S. Č. Živković-Zlatanović, D. S. Djordjević, R. E. Harte: Ruston, Riesz and perturbation classes, *J. Math. Anal. Appl.* **389** (2012), 871–886.

Theorem

If $T : A \rightarrow B$ satisfies the strong Riesz property and $a \in A$, then

$$\eta\sigma(Ta, B) = \eta\omega_T(a, A) = \eta\omega_T^{\text{comm}}(a, A).$$

Theorem

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S. Č. Živković-Zlatanović, R. E. Harte: Polynomially Riesz elements, submitted.

THANK YOU