Factorizable completely positive maps and the Connes embedding problem

Joint work with Uffe Haagerup, and part in collaboration with

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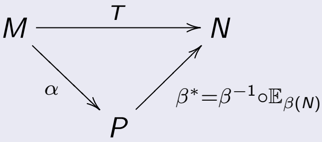
Outline

1. Factorizable maps and the Asymptotic Quantum Birkhoff Conjecture
2. Extreme points and factorizability
3. Remarks on the semigroup case
4. Asymptotic properties of factorizable maps and the Connes embedding problem
5. Holevo–Werner channels

Definition (Anantharaman-Delaroche, 2005)

- Let $(M\,,\phi)\,,(N\,,\psi)$ be vN algebras with n.f. tracial states.

A trace-preserving UCP map $T\colon M \to N$ is called *factorizable* if $\exists$ vN algebra P with n.f. tracial state χ and injective trace-preserving unital $*$-homs $\alpha\colon M \to P, \beta\colon N \to P$ s.t. $T = \beta^* \circ \alpha$.



- Markov maps between abelian vN algebras are factorizable.

- The set of factorizable maps $\mathcal{F}(M, N)$ is convex and closed under composition and taking adjoints. In particular, for $n \geq 2$,

$$\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))) \subseteq \mathcal{F}(M_n(\mathbb{C})) \subseteq \mathrm{UCPT}_n\,.$$

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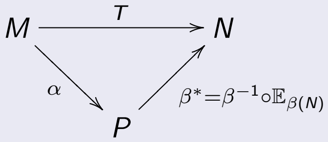

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Theorem (Anantharaman-Delaroche, 2005)

$T : M \to M$ factorizable with $T = T^* \implies T^2$ has a Rota dilation.

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Connections to quantum information

- **Gregoratti-Werner** (2003): Channels which are convex combinations of unitarily implemented ones allow for complete error correction, given suitable feedback of classical information from the environment.
- **Kümmerer** (1983): $\mathrm{UCPT}_2 = \mathrm{conv}(\mathrm{Aut}(M_2(\mathbb{C})))$.
- For $n \geq 3$: $\mathrm{UCPT}_n \supsetneq \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$

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Let $T \in UCPT_n$, $n \geq 3$. Then T satisfies the following asymptotic quantum Birkhoff property *(**AQBP**):*

$$\lim_{k\to\infty} d_{cb}\left(\bigotimes_{i=1}^{k} T\,,\, conv(Aut(\bigotimes_{i=1}^{k} M_n(\mathbb{C})))\right) = 0\,.$$

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Mendl-Wolf (2009): $\exists\, T \in \mathrm{UCPT}_3$ s.t. $T \notin \mathrm{conv}(\mathrm{Aut}(M_3(\mathbb{C})))$, but

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Theorem (Haagerup-M, 2011)

Let $T \in UCPT_n$, where $n \geq 3$. Then, for all $k \geq 1$,

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- If T is not factorizable, then $d_{\mathrm{cb}}(T, \mathcal{F}(M_n(\mathbb{C}))) > 0$, as $\mathcal{F}(M_n(\mathbb{C}))$ is norm-closed. Since

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TFAE for $T \in UCPT_n$, $n \geq 3$, written in Choi canonical form

$$Tx = \sum_{i=1}^{d} a_i^* x a_i \,, \quad x \in M_n(\mathbb{C})\,.$$

1) *T is factorizable*

2) *$\exists$ vN algebra N with nf tracial state τ_N and $u \in \mathcal{U}(M_n(N))$ st*

$$Tx = (id_{M_n(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u)\,, \quad x \in M_n(\mathbb{C})\,.$$

We say that T has an exact factorization through $M_n(\mathbb{C}) \otimes N$.

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$$u := \sum_{i=1}^{d} a_i \otimes v_i \in \mathcal{U}(M_n(N)), \quad \tau_N(v_i^* v_j) = \delta_{ij}\,, 1 \leq i,j \leq d$$

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Factorizable maps are obtained by coupling the input system to a maximally mixed ancillary one, executing a unitary rotation on the combined system, and tracing out the ancilla.

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Examples of non-factorizable maps

Corollary

Let $T \in UCPT_n$, with canonical form $T = \sum_{i=1}^{d} a_i^* x a_i$. If $d \geq 2$ and the set

$$\{a_i^* a_j : 1 \leq i, j \leq d\}$$

is linearly independent, then T is not factorizable.

Proof: Assume T factorizable. Then $\exists (N, \tau_N)$, $\exists v_1, \ldots, v_d \in N$ st

$$u := \sum a_i \otimes v_i \in \mathcal{U}(M_n(\mathbb{C}) \otimes N), \qquad \tau_N(v_i^* v_j) = \delta_{ij}.$$

$$\Longrightarrow \sum_{i,j=1}^{d} a_i^* a_j \otimes (v_i^* v_j - \delta_{ij} 1_N) = u^* u - \left(\sum_{i=1}^{d} a_i^* a_i \right) \otimes 1_N = 0\,.$$

By the linear independence of the set $\{a_i^* a_j : 1 \leq i, j \leq d\}$,

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In particular (since $d \geq 2$), $v_1^* v_1 = v_2^* v_2 = 1_N$ and $v_1^* v_2 = 0_N$. Impossible, since N is finite. $\lightning$

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Example

As an application of corollary, by letting

$$a_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad a_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$a_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

we obtained a (first example of a) non-factorizable unital channel. Turned out to be the Holevo-Werner channel, W_3^-, in dimension $n = 3$.

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Choi (1975): $T \in \partial_e(\mathrm{UCP}_n)$ if and only if $\{a_i^* a_j : 1 \leq i, j \leq d\}$ is a linearly independent set.

By the corollary, if

$$T \in \partial_e(\mathrm{UCP}_n) \cap \mathrm{UCPT}_n$$

then T is not factorizable. (Hence T does not satisfy AQBP.)

Crann-Neufang (2012): A class of unital quantum channels arising from abstract harmonic analysis failing the AQBP:

Given G finite group, ϕ pos. definite function on G with $\phi(e) = 1$

$\rightsquigarrow B_\phi = (\phi(t^{-1}s))_{t,s \in G} \in M_{|G|}(\mathbb{C}) \rightsquigarrow T_{B_\phi}$ unital Schur channel.

Conditions ensuring that T_{B_ϕ} is an extreme point are discussed.

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Theorem (Haagerup-M.-Ruskai)

Let $n \geq 3$ and let $S\colon \mathbb{C}^n \to \mathbb{C}^n$ be the cyclic shift.

(1) *Let $U_1, \dots, U_n \in \mathcal{U}(n-1)$ and set*

$$a_i = \frac{1}{\sqrt{n-1}} S^i \begin{pmatrix} U_i & 0 \\ 0 & 0 \end{pmatrix} S^{-i}, \quad 1 \leq i \leq n\,.$$

Set $Tx = \sum_{i=1}^n a_i^* x a_i$, $x \in M_n(\mathbb{C})$. Then $T \in UCPT_n$ and with probability one (w.r.t. Haar measure on $\prod_{i=1}^n \mathcal{U}(n-1)$),

$$T \in \partial_e(UCP_n) \cap \partial_e(CPT_n)\,.$$

In particular, T is not factorizable.

(2) *Same conclusion holds for*

$$a_i = \frac{1}{\sqrt{n-1+t^2}} S^i \begin{pmatrix} U_i & 0 \\ 0 & t \end{pmatrix} S^{-i}, \quad 1 \leq i \leq n\,,$$

where $t > 0$, $t \neq 1$ (fixed).

Landau and Streater (1993): $T \in \partial_e(\mathrm{UCPT}_n)$ if and only if

$$\{a_i^* a_j \oplus a_j a_i^* : 1 \leq i, j \leq d\}$$

is a linearly independent set. Hence

$$\partial_e(\mathrm{UCPT}_n) \supseteq \big(\partial_e(\mathrm{UCP}_n) \cup \partial_e(\mathrm{CPT}_n)\big) \cap \mathrm{UCPT}_n.$$

Mendl-Wolf (2009): above inclusion is strict for $n = 3$.

Ohno (2010): concrete examples for $n = 3$, $n = 4$.

Further examples (motivated by a question of Farenick, 2010):
Haagerup-M.-Ruskai: A family $(T_t)_{t\in[0,1]} \subset \mathrm{UCPT}_3$ s.t.

$$T_t \in \partial_e(\mathrm{UCPT}_3) \setminus \big(\partial_e(\mathrm{UCP}_3) \cup \partial_e(\mathrm{CPT}_3)\big), \quad t \in (0,1) \setminus \{1/2\}.$$

Moreover, T_t is factorizable, $0 \leq t \leq 1$, (through $M_3(\mathbb{C}) \otimes M_2(\mathbb{C})$).

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Remarks on the semigroup case

Kümmerer-Maassen (1987): For $n \geq 3$, if $(T_t)_{t\geq 0}$ is a one-parameter semigroup of <u>self-adjoint</u> UCPT_n maps, then

$$T_t \in \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C}))), \quad t \geq 0.$$

In particular,

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Haagerup-M. (2011): Example of a semigroup $(T_t)_{t\geq 0}$ of <u>non self-adjoint</u> UCPT_4 maps for which $\exists\ t_0 > 0$ s.t. T_t is <u>not</u> factorizable, for any $0 < t < t_0$.

Junge-Ricard-Shlyakhtenko (independently, **Dabrowsky**) (2012): If $(T_t)_{t\geq 0}$ is a strongly continuous semigroup of self-adjoint UCPT maps on a finite von Neumann algebra, then

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Kümmerer-Maassen (1987): For $n \geq 3$, if $(T_t)_{t\geq 0}$ is a one-parameter semigroup of self-adjoint UCPT_n maps, then

$$T_t \in \text{conv}(\text{Aut}(M_n(\mathbb{C}))), \quad t \geq 0.$$

In particular,

$$T_t \text{ is factorizable}, \quad t \geq 0.$$

Haagerup-M. (2011): Example of a semigroup $(T_t)_{t\geq 0}$ of non self-adjoint UCPT_4 maps for which $\exists\ t_0 > 0$ s.t. T_t is not factorizable, for any $0 < t < t_0$.

Junge-Ricard-Shlyakhtenko (independently, **Dabrowsky**) (2012): If $(T_t)_{t\geq 0}$ is a strongly continuous semigroup of self-adjoint UCPT maps on a finite von Neumann algebra, then

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2. Extreme points and factorizability
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Natural question: Does every factorizable unital quantum channel satisfy the AQBP?

This question (that arose at a 2010 Banff meeting) gained a lot of interest due to the following connection to the *Connes embedding problem*, whether every II_1-factor (on a separable Hilbert space) embeds in an ultrapower R^ω of the hyperfinite II_1 factor R:

Theorem (Haagerup-M, 2011)

If for any $n \geq 3$, every factorizable $UCPT_n$ map satisfies the AQBP, then the Connes embedding problem has a positive answer.

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If for any $n \geq 3$, every factorizable $UCPT_n$ map satisfies the AQBP, then the Connes embedding problem has a positive answer.

Example

Let $\beta = 1/\sqrt{5}$ and set

$$B := \begin{pmatrix} 1 & \beta & \beta & \beta & \beta & \beta \\ \beta & 1 & \beta & -\beta & -\beta & \beta \\ \beta & \beta & 1 & \beta & -\beta & -\beta \\ \beta & -\beta & \beta & 1 & \beta & -\beta \\ \beta & -\beta & -\beta & \beta & 1 & \beta \\ \beta & \beta & -\beta & -\beta & \beta & 1 \end{pmatrix}.$$

The unital Schur channel T_B is factorizable, but

$$T_B \notin \operatorname{conv}(\operatorname{Aut}(M_6(\mathbb{C})))$$

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Does T_B satisfy the AQBP?

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Theorem (Haagerup-M)

Let T be a unital Schur channel on $M_n(\mathbb{C})$ and S be a unital Schur channel on $M_k(\mathbb{C})$, $k, n \geq 2$. Then

$$d_{cb}\Big(T \otimes S, conv(Aut(M_{nk}(\mathbb{C}))\Big) \geq \frac{1}{2}\, d_{cb}\Big(T, conv(Aut(M_n(\mathbb{C}))\Big)$$

In particular, if $T \notin conv(Aut(M_n(\mathbb{C})))$, then T fails the AQBP.

Conclusion: The factorizable map T_B above fails the AQBP.

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Connection to the Connes embedding problem

Theorem (Haagerup-M)

Let $T \in UCPT_n$ be factorizable. TFAE:

(1) *T has an exact factorization through a finite von Neumann algebra which embeds into R^ω, i.e.,*

$$\exists\ (N, \tau_N) \hookrightarrow R^\omega\,, \quad \exists\ u \in \mathcal{U}(M_n(N)) \text{ s.t.}$$

$$Tx = (id_{M_n(\mathbb{C})} \otimes \tau_N)\big(u^*(x \otimes 1_N)u\big)\,, \quad x \in M_n(\mathbb{C})\,.$$

(2) *T admits an approximate factorization through matrix algebras.*

(3) $\lim_{k\to\infty} d_{cb}\Big(T \otimes S_k\,, conv\big(Aut(M_n(\mathbb{C}) \otimes M_k(\mathbb{C}))\big)\Big) = 0,$
where S_k is the completely depolarizing channel:

$$S_k(y) = \tau_k(y)1_k, \quad y \in M_k(\mathbb{C}).$$

Theorem (Haagerup-M)

The Connes embedding problem has a positive answer if and only if every factorizable $UCPT_n$ map satisfies one of the equivalent conditions in previous theorem, for all $n \geq 3$.

Ideas of proof of previous theorem:

- If T has an exact factorization through $M_n(\mathbb{C}) \otimes M_k(\mathbb{C})$, then

$$T \otimes S_k \in \mathrm{conv}\big(\mathrm{Aut}\big(M_n(\mathbb{C}) \otimes M_k(\mathbb{C})\big)\big).$$

- If $T \otimes S_k \in \mathrm{conv}\big(\mathrm{Aut}(M_n(\mathbb{C}) \otimes M_k(\mathbb{C}))\big)$, then T has an exact factorization through

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Idea of proof: $(\Leftarrow)$ **Dykema-Jushenko** (2009):

$$\mathcal{F}_n : \; = \overline{\bigcup_{k\geq 1} \left\{ B = (b_{ij}) \in M_n(\mathbb{C}) : b_{ij} = \tau_k(u_i u_j^*)\,, u_j \in \mathcal{U}(M_k(\mathbb{C})) \right\}}$$

$$\mathcal{G}_n : \; = \Big\{ B = (b_{ij}) \in M_n(\mathbb{C}) : b_{ij} = \tau_M(u_i u_j^*)\,, u_j \in \mathcal{U}(M)\,, \text{ for some vN algebra } (M, \tau_M) \text{ with n.f. tracial state} \Big\}$$

$$= \{ B \in M_n(\mathbb{C}) : \text{ Schur multiplier } T_B \text{ is factorizable} \}.$$

By **Kirchberg** (1993): The Connes embedding problem has a positive answer iff $\mathcal{F}_n = \mathcal{G}_n$, for all $n \geq 1$.

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Idea of proof — continued:

Assume that the Connes embedding problem has a negative answer. Then $\mathcal{G}_n \setminus \mathcal{F}_n \neq \emptyset$, for some $n \geq 1$. Choose

$$B = (b_{ij})_{i,j=1}^{n} \in \mathcal{G}_n \setminus \mathcal{F}_n\,.$$

Then the Schur multiplier T_B has an exact factorization through a finite vN algebra embeddable into R^ω, so $\exists u_1, \ldots u_n \in \mathcal{U}(R^\omega)$ s.t.

$$b_{ij} = \tau_{R^\omega}(u_i^* u_j), \quad 1 \leq i, j \leq n\,.$$

Approximate each b_{ij} by $\tau_R(v_i^* v_j)$, where $v_i \in \mathcal{U}(R)$, and further by unitary matrices (via Kaplansky). Hence B can be approximated by a sequence B_k whose Schur multiplier T_{B_k} admits an exact factorization through a matrix algebra. This implies $B \in \mathcal{F}_n$ $\lightning$.

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Some concrete examples

Let $n \geq 2$. Consider the *Holevo-Werner channel* in dimension n:

$$W_n^-(x) = \frac{1}{n-1}\left(\mathrm{Tr}(x)1_n - x^t \right), \quad x \in M_n(\mathbb{C}).$$

It has an analogue

$$W_n^+(x) = \frac{1}{n+1}\left(\mathrm{Tr}(x)1_n + x^t \right), \quad x \in M_n(\mathbb{C}).$$

- $W_n^-, W_n^+ \in \mathrm{UCPT}_n$.
- $S_n \in \mathrm{conv}\{W_n^-, W_n^+\}$, since

$$S_n(x) = \frac{1}{n}\mathrm{Tr}(x)1_n = \frac{n-1}{2n}W_n^-(x) + \frac{n+1}{2n}W_n^+(x), \ x \in M_n(\mathbb{C}).$$

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Denote

$$a_{ij} = e_{ij} - e_{ji}, \quad b_{ij} = e_{ij} + e_{ji}\,, \quad 1 \leq i,j \leq n$$

- For all $x \in M_n(\mathbb{C})$ we have (Choi canonical forms):

$$W_n^-(x) = \frac{1}{n-1}\sum_{i<j} a_{ij}\, x\, a_{ij}^*$$

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- W_3^- is not factorizable.

Theorem (Mendl-Wolf, 2009)

(1) $W_n^+ \in conv(Aut(M_n(\mathbb{C})))$, *for all* $n \geq 2$.

(2) $W_n^- \in conv(Aut(M_n(\mathbb{C})))$, *for all* n *even*.

(3) *For* n *odd and* $0 \leq \lambda \leq 1$,

$$\lambda\, W_n^+ + (1-\lambda)\, W_n^- \in conv\big(Aut(M_n(\mathbb{C}))\big) \iff \lambda \geq \frac{1}{n}.$$

In particular, $W_n^- \notin conv\big(Aut(M_n(\mathbb{C}))\big)$.

Theorem (Haagerup-M)

(1) $d_{cb}(W_3^-, \mathcal{F}(M_3(\mathbb{C}))) = \frac{4}{27}$.

(2) *For* n *odd,* $n \neq 3$,

W_n^- *has an exact factorization through* $M_n(\mathbb{C}) \otimes M_4(\mathbb{C})$.

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In each such case, $\lambda\, W_3^+ + (1-\lambda)\, W_3^-$ *has an exact factorization through* $M_3(\mathbb{C}) \otimes M_3(\mathbb{C})$.

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For $0 \leq \lambda \leq 1$, set $T_\lambda := \lambda\, W_3^+ + (1-\lambda)\, W_3^-$. By **Mendl-Wolf**:

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Theorem (Mendl-Wolf, 2009)

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There exists $\lambda_1 \in \big(0, \tfrac{1}{3}\big)$ such that for all $\lambda \geq \lambda_1$,

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About some of the proofs

- Proof of the fact that W_5^- is factorizable.

$$W_5^- = \frac{1}{4}\sum_{i<j} a_{ij} x a_{ij}^*, \quad a_{ij} = e_{ij} - e_{ji} \,,$$

W_5^{-1} is factorizable iff $\exists\,(N, \tau_N)$ and $u = (u_{ij})_{i,j=\overline{1,5}} \in \mathcal{U}(M_n(N))$ s.t. $u_{ij} = -u_{ji}$, $\forall i,j$ and $\{u_{ij}\}_{i,j=\overline{1,5}}$ is orthonormal w.r.t. inner product given by τ_N.

This can be achieved with $(N, \tau_N) = (M_4(\mathbb{C}), \mathrm{tr}_4)$.

Key: $\exists v_1, \ldots, v_5 \in \mathcal{U}(M_4(\mathbb{C}))$ self-adjoint anti-commuting which form an orthonormal set w.r.t. inner product given by tr_4.

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Let $a = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, b = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$. Then

$$\sigma = \begin{pmatrix} 0 & a & b & b & a \\ a & 0 & a & b & b \\ b & a & 0 & a & b \\ b & b & a & 0 & a \\ a & b & b & a & 0 \end{pmatrix} \in \mathcal{U}(M_5(\mathbb{C}))\,.$$

Set $u\colon = \begin{pmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_5 \end{pmatrix} (\sigma \otimes 1_4) \begin{pmatrix} v_1 & & 0 \\ & \ddots & \\ 0 & & v_5 \end{pmatrix} = (\sigma_{ij} v_i v_j)_{i,j=\overline{1,5}}\,.$

Then $u \in \mathcal{U}(M_5(\mathbb{C}) \otimes M_4(\mathbb{C}))$ does the trick.

- Proof of the fact that $T = \frac{2}{27} W_3^+ + \frac{25}{27} W_3^-$ is factorizable.

One can check that for $x \in M_3(\mathbb{C})$,

$$\left(\frac{2}{27} W_3^+ + \frac{25}{27} W_3^-\right)(x) = (\mathrm{id}_{M_3(\mathbb{C})} \otimes \mathrm{tr}_3)(u(x \otimes 1_3)u^*),$$

where

$$u = \begin{pmatrix} \frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{2}{3} & 0 & 0 & 0 & -\frac{2}{3} & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \in \mathcal{U}(M_9(\mathbb{C}))\,.$$

Then

$$
\begin{aligned}
d_{\mathrm{cb}}(W_3^-, \mathcal{F}(M_3(\mathbb{C}))) &\leq \left\| W_3^- - \left(\frac{2}{27} W_3^+ + \frac{25}{27} W_3^- \right) \right\|_{\mathrm{cb}} \\
&= \frac{2}{27} \| W_3^+ - W_3^- \|_{\mathrm{cb}} = \frac{4}{27},
\end{aligned}
$$

wherein we have used the fact that

$$\| W_n^+ - W_n^- \|_{\mathrm{cb}} = 2, \quad n \geq 2 .$$

The statement that

- $\lambda W_3^+ + (1-\lambda) W_3^-$ is factorizable if and only if $\frac{2}{27} \leq \lambda \leq 1$

follows once we show that $d_{\mathrm{cb}}(W_3^-, \mathcal{F}(M_3(\mathbb{C}))) \geq 4/27$, since

$$2\lambda = \| W_3^- - (\lambda W_3^+ + (1-\lambda) W_3^-) \|_{\mathrm{cb}} \geq d_{\mathrm{cb}}(W_3^-, \mathcal{F}(M_3(\mathbb{C}))) = 4/27 .$$

Note that the ('if') part follows right-away by convexity.

Then

$$\begin{aligned} d_{\mathrm{cb}}(W_3^-, \mathcal{F}(M_3(\mathbb{C}))) &\leq \left\| W_3^- - \left(\frac{2}{27} W_3^+ + \frac{25}{27} W_3^- \right) \right\|_{\mathrm{cb}} \\ &= \frac{2}{27} \| W_3^+ - W_3^- \|_{\mathrm{cb}} = \frac{4}{27}, \end{aligned}$$

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An averaging technique (twirl)

For $T \in \mathcal{B}(M_n(\mathbb{C}))$ set

$$F(T) := \int_{\mathcal{U}(n)} \mathrm{ad}(u)\, T\, \mathrm{ad}(u^t)\, du\,,$$

where du is the Haar measure on $\mathcal{U}(n)$.

Properties:

- If $T \in \mathrm{UCPT}_n$ then $F(T) \in \mathrm{UCPT}_n$. Moreover,

$$\|F(T)\|_{\mathrm{cb}} \leq \|T\|_{\mathrm{cb}}\,.$$

- $F(\mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))) \subseteq \mathrm{conv}(\mathrm{Aut}(M_n(\mathbb{C})))$.
- $F(\mathcal{F}(M_n(\mathbb{C}))) \subseteq \mathcal{F}(M_n(\mathbb{C}))$.

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Choi (1975): $T \in \mathcal{B}(M_n(\mathbb{C}))$ is CP $\iff$ $\hat{T}$ is positive, where

$$\hat{T} := \frac{1}{n} \sum_{i,j=1}^{n} T(e_{ij}) \otimes e_{ij} \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) .$$

Vollbrecht-Werner (2001): $\widehat{F(T)} = E(\hat{T})$, where

$$E(x) = \int_{\mathcal{U}(n)} (u \otimes u) x (u^* \otimes u^*) du, \quad x \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C}),$$

is the trace-preserving cond. expectation of $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ onto $\text{span}\{P^+, P^-\}$, where P^+, P^- are the orthogonal projections onto $(\mathbb{C}^n \otimes \mathbb{C}^n)_{\text{sym}}$ and $(\mathbb{C}^n \otimes \mathbb{C}^n)_{\text{antisym}}$, respectively.

- $F(W_n^+) = W_n^+$ and $F(W_n^-) = W_n^-$.

Theorem (Haagerup-M)

If $T \in UCPT_n$, then

$$F(T) \in conv\{W_n^+, W_n^-\}.$$

More precisely, if $T = \sum_{i=1}^d a_i x a_i^*$ (Choi canonical form), then

$$F(T) = c^+(T) W_n^+ + c^-(T) W_n^-,$$

where $c^+(T) = \frac{1}{4}\sum_{i=1}^{d} \|a_i + a_i^t\|_2^2$, $c^-(T) = \frac{1}{4}\sum_{i=1}^{d} \|a_i - a_i^t\|_2^2$.

Corollary:

(1) If $a_i^t = a_i$ for all i, then $F(T) = W_n^+$.

(2) If $a_i^t = -a_i$ for all i, then $F(T) = W_n^-$.

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- Proof of the fact that W_n^- is factorizable, for all $n \geq 7$, n odd.

Let $k = \frac{n-5}{2} \in \mathbb{N}$. Define $S \in \mathrm{UCPT}_n$ by

$$S(x) := \begin{pmatrix} W_5^-(x_{11}) & 0 \\ 0 & W_{2k}^-(x_{22}) \end{pmatrix},$$

where $x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in \mathbb{C}^n = \mathbb{C}^5 \oplus \mathbb{C}^{2k}$.

$$W_5^- \in \mathcal{F}(M_5(\mathbb{C})), W_{2k}^- \in \mathrm{conv}(\mathrm{Aut}(M_{2k}(\mathbb{C}))) \Rightarrow S \in \mathcal{F}(M_n(\mathbb{C})).$$

An application of above corollary shows that $F(S) = W_n^-$. The conclusion follows.

- Proof of the fact that $d_{\mathrm{cb}}(W_3^-, \mathcal{F}(M_3(\mathbb{C}))) \geq 4/27$.

Let $T \in \mathcal{F}(M_3(\mathbb{C}))$. Then $\exists (N, \tau_N)$ and $u \in \mathcal{U}(M_3(N))$ s.t.

$$Tx = (\mathrm{id}_{M_3(\mathbb{C})} \otimes \tau_N)(u^*(x \otimes 1_N)u), \quad x \in M_3(\mathbb{C}).$$

A refinement of previous theorem shows that

$$F(T) = \lambda W_3^+ + (1-\lambda) W_3^-,$$

where $\lambda = \frac{1}{4}\|u + u^t\|_2^2$. Moreover, we can prove that $\lambda \geq \frac{2}{27}$. Then

$$\|W_3^- - T\|_{\mathrm{cb}} \geq \|W_3^- - F(T)\|_{\mathrm{cb}} = \lambda \|W_3^- - W_3^+\|_{\mathrm{cb}} = 2\lambda \geq 4/27.$$