$*$-Algebras generated by projections and their representations

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$*$-Algebras generated by projections and families of orthoprojections

Let $\mathcal{P}_n$ be a $*$-algebra generated by n self-adjoint idempotents:
$p_1, \ldots, p_n$, $p_j^* = p_j = p_j^2$, $j = 1, \ldots, n$
A representation of $\mathcal{P}_n$ is determined by a collection P_j, $j = 1, \ldots, n$ of orthoprojections on some Hilbert space H
Our task is to describe representations of $\mathcal{P}_n$, i.e., classify n-tuples of projections.

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As we will see, for $n > 2$ this problem appears too complicated, and we apply extra conditions on the set of projections, as a rule in the form of algebraic relations between the generators:

$$f_k(p_1, \ldots, p_n) = 0, \quad k = 1, \ldots, m$$

where f_k are some polynomials.

Systems of subspaces of a Hilbert space

Definition

Let $H_j \subset H$, $j = 1, \dots, n$, be closed subspaces of a Hilbert space H. We write

$$S = (H; H_1, \dots, H_n)$$

and say that S is a system of subspaces in H.

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For a family of projections P_j, $j = 1, \dots, n$ define $H_j = \operatorname{Im} P_j$, then any representation of $\mathcal{P}_n$ defines a system of subspaces and vice versa, therefore,

the problem of description of systems of subspaces is equivalent to the description of representations of $\mathcal{P}_n$.

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Our task is to classify indecomposable systems of subspaces up to unitary equivalence = classify irreducible representations of $\mathcal{P}_n$ up to unitary equivalence.

$*$-Tame and $*$-wild problems

In representation theory, some problems have nice explicit solution, while other ones are extremely complicated. E.g., any orthoprojection P up to unitary equivalence is uniquely determined by the dimension and co-dimension of $\mathrm{Im} P$. On the other hand, there is no satisfactory description for a pair of bounded self-adjoint operators A, B in a Hilbert space H. Moreover, the latter problem contains a subproblem of desctiption of any collections of finite or even countable number of self-adjoint operators.

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$*$-Finite problem: there exist only finitely many unitary inequivalent irreducible representations.
$*$-Tame problem: one can present an explicit list of all, up to unitary equivalence, irreducible representations.
$*$-Wild problem: the problem contains the description of pairs of self-adjoint operators.

Example of $*$-wild problem

Theorem

Description, up to unitary equivalnece, of all pairs (P, Q) of idempotents in a Hilbert space H is a $*$-wild problem.

Proof.

Let A, B be bounded self-adjoint operators in H', let $H = H' \oplus H'$. Consider the idempotents in H of the form

$$P = \begin{pmatrix} I & A + iB \\ 0 & 0 \end{pmatrix}, \quad Q = \frac{1}{2} \begin{pmatrix} I & I \\ I & I \end{pmatrix}$$

Then the pair (P, Q) in H is irreducible iff the pair (A, B) is irreducible in H'. Two pairs of such form, (P, Q), and (P', Q') are unitary equivalent in H iff the corresponding pairs (A, B) and (A', B') are unitary equivalent in H'. □

Single projection

Description of representations of $\mathcal{P}_1$ is $*$-finite problem.

Any representations of $\mathcal{P}_1$ is determined by a single projection P which is uniquely determined by dimension and co-dimension of its image $\operatorname{Im} P$.
All irreducible representations are one-dimensional:

- $H = \mathbb{C}$, $P = 0$,
- $H = \mathbb{C}$, $P = 1$.

For any projection P, the space H can be uniquely decomposed into invariant w.r.t. P direct sum $H = H_0 \oplus H_1$ so that $P|_{H_0} = 0$ and $P|_{H_1} = I$.

Pair of projections. Irreducible representations

The problem of unitary description of representations of $\mathcal{P}_2$ is tame.

Theorem

Any irreducible representation of $\mathcal{P}_2$ has dimension 1 or 2. All irreducible representations, up to unitary equivalence, are the following.

- *Four one-dimensional, $H = \mathbb{C}$, $P_1, P_2 \in \{0, 1\}$.*

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- *Four one-dimensional, $H = \mathbb{C}$, $P_1, P_2 \in \{0, 1\}$.*
- *One-parameter series of two-dimensional, $H = \mathbb{C}^2$,*

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix},$$

$0 < c < 1$, $s = \sqrt{1 - c^2}$ *(general position representations).*

Pair of projections. Structure theorem

Theorem

Let P_1, P_2 be projections in a Hilbert space H. Then H uniquely decomposes into direct sum of invariant w.r.t. P_1 and P_2 subspaces,
$H = H_{00} \oplus H_{01} \oplus H_{10} \oplus H_{11} \oplus \mathbb{C}^2 \otimes H_+$,
so that in H_{jk} $P_1 = jI$, $P_2 = kI$, $j, k \in \{0, 1\}$, and in $\mathbb{C}^2 \otimes H_+$

$$P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where C is a self-adjoint operator in H_+, $0 < C < I$, $S = \sqrt{1 - C^2}$.

We say that the projections P_1, P_2 are in general position if $H_{01} = H_{10} = H_{11} = 0$.

Pair of projections. Angles between subspaces

Given a general position pair of projections in $\mathbb{C}^2$,

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} c^2 & cs \\ cs & s^2 \end{pmatrix},$$

the image of P_1 is spanned by the vector $v_1 = (1, 0)$ and the image of P_2 is spanned by the vector $v_2 = (c, s)$, thus $c = \cos\phi$, where ϕ is the angle between v_1 and v_2

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The structure theorem states that the general position part splits into (discrete of continuous) direct sum of invariant 2-dimensional planes, such that intersection of each of the subspaces, H_1, H_2 with any plane is a line, with the angle between these lines determined by the corresponding point of $\sigma(C)$.

Angles between subspaces (continued)

Definition

We say that angles between subspaces H_1, H_2 are in set $\{\phi_1, \ldots, \phi_m\}$ if the corresponding projections P_1 and P_2 are in general position and $\sigma(C) \subset \{\cos\phi_1, \ldots, \cos\phi_m\}$.

Angles between subspaces (continued)

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If there is only one angle between H_1 and H_2, i.e. $\sigma(C^2) = \tau \in (0,1)$, then

$$P_1 = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} \tau I & \sqrt{\tau(1-\tau)} I \\ \sqrt{\tau(1-\tau)} I & (1-\tau) I \end{pmatrix},$$

and the projections satisfy

$$P_1P_2P_1 = \tau P_1, \quad P_2P_1P_2 = \tau P_2. \tag{1}$$

Conversely, (1) implies that P_1 and P_2 are in general position and $\sigma(C^2) = \tau$.

Angles between subspaces (continued)

Also, P_1 and P_2 are in general position with angles in $\{\phi_1, \ldots, \phi_m\}$ iff

$$\prod_{k=1}^{m}(P_1 P_2 P_1 - \tau_k P_1) = 0, \quad \prod_{k=1}^{m}(P_2 P_1 P_2 - \tau_k P_2) = 0.$$

where $\tau_k = \cos^2 \phi_k$, $k = 1, \ldots, m$.
Algebras generated by families of projections with relations of such sort were introduced and studied in [N.Popova, A.Strelets, Yu.Samoilenko, 2007-2009]

Triples of projections. Wildness

Theorem (S.A.Kruglyak, Yu.S.Samoilenko, 1980)

The problem of unitary description of representations of $\mathcal{P}_3$ is $*$-wild.

To prove this, one can take a pair (U, V) of unitary operators in H and explicitly construct three projections, P_1, P_2, P_3 in $\tilde{H} = \mathbb{C}^4 \otimes H$ as block matrices whose matix units are expressed via U and V in such a way that the construction preserves irreducibility and unitary equivalence.

Operator Gram matrix. Construction

Let $P_1, \ldots, P_n$ be a family of projections in H, let $H_j = \operatorname{Im} P_j$, $j = 1, \ldots, n$. Let $S_j \colon H_j \to H$ be isometric embeddings, so that $S_j S_j^* = P_j$, $S_j^* S_j = I_{H_j}$. Consider space $\tilde{H} = H_1 \oplus \cdots \oplus H_n$ and operator $J = (S_1, \ldots, S_n) \colon \tilde{H} \to H$.

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Definition (Yu.Samoilenko, A.Strelets, 2009; I.Feschenko, A.Strelets, 2012)

Operator $G = J^* J \colon \tilde{H} \to \tilde{H}$ is called operator Gram matrix, corresponding to the system of subspaces $(H; H_1, \dots, H_n)$.

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Block entries of the operator Gram matrix are $(S_j^* S_k)_{j,k=1}^n$, therefore in the case where all P_j are one-dimensional projections we have $H_j = \mathbb{C}\langle e_j \rangle$, $\|e_j\| = 1$ and G is the Gram matrix of the system of vectors $(e_1, \dots, e_n)$.

Operator Gram matrix. Properties

Theorem (I.Feschenko, A.Strelets, 2012)

Operator Gram matrix possesses the following properties.

1. $G = G^*$, $G \geq 0$
2. *Diagonal entries of* G *are identity operators,* $G_{jj} = I_{H_j}$, $j = 1, \ldots, n$.
3. $G_{jk} = 0 \iff H_j \perp H_k$.
4. H_j *and* H_k *are in general position with set of angles* $(\phi_1, \ldots, \phi_m)$ *iff* $\sigma(G_{jk}G_{kj}) = \sigma(G_{kj}G_{jk}) \subset \{\tau_1, \ldots, \tau_m\}$, $\tau_p = \cos^2 \phi_p$, $p = 1, \ldots, m$.
5. $\sum_j \alpha_j P_j = I$ *for some* $\alpha_j > 0$, $j = 1, \ldots, n$, *iff* DGD *is a projection,* $D = diag(\sqrt{\alpha_1} I_{H_1}, \ldots, \sqrt{\alpha_n} I_{H_n})$.

Operator Gram matrix. Properties (continued)

Let $Q_1, \ldots, Q_n$ be the projections on H_j in $\tilde{H}$.

Theorem (I.Feschenko, A.Strelets, 2012)

- *Family* $(P_1, \ldots, P_n)$ *in* H *is irreducible iff the family* $(G, Q_1, \ldots, Q_n)$ *is irreducible in* $\tilde{H}$*.*
- *Families* $(P_1, \ldots, P_n)$ *and* $(P'_1, \ldots, P'_n)$ *are unitary equivalent iff the corresponding families* $(G, Q_1, \ldots, Q_n)$ *and* $(G', Q'_1, \ldots, Q'_n)$ *are unitary equivalent.*

Inverse construction

Assume we have projections $Q_1, \ldots, Q_n$, in a Hilbert space $\tilde{H}$, $\sum_{k=1}^{n} Q_k = I$, and bounded $B \geq 0$ in $\tilde{H}$ such that $Q_j B Q_j = Q_j$, $j = 1, \ldots, n$. Is it possible to counstruct a family $P_1, \ldots, P_n$, for which B would be the Gram operator?

Inverse construction (continued)

Let H' be the closure of $\operatorname{Im} B$ and let $S\colon H' \to \tilde{H}$ be isometric embedding. Define $J = S^*\sqrt{B}\colon \tilde{H} \to H'$. Obviously, $J^*J = B$. Put $P_j' = JQ_jJ^*\colon H' \to H'$, $j = 1, \ldots, n$.

Inverse construction (continued)

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Theorem (I.Feschenko, A.Strelets, 2012)

- *P'_j, $j = 1, \ldots, n$ are projections.*
- *Let B be the Gram operator of some family $(P_1, \ldots, P_n)$. The constructed above family $(P'_1, \ldots, P'_n)$ is unitary equivalent to $(P_1, \ldots, P_n)$.*
- *Let G' be the Gram operator of the constructed family P'_j, $j = 1, \ldots, n$, and $Q'_1, \ldots, Q'_n$ are the corresponding projections. Then the family $(B, Q_1, \ldots, Q_n)$ is unitary equivalent to $(G', Q'_1, \ldots, Q'_n)$.*

Class of problems

Condition on the collection: each pair of subspaces are orthogonal or angles between them are in a fixed finite set.

$S = (H, H_1, \dots, H_n)$

$T_{jk} = \{0 < \tau_{jk}^{(1)} < \dots < \tau_{jk}^{(m_{jk})} < 1\}$, $\tau_{jk} = \cos^2 \phi_{jk}$,
$j, k = 1, \dots, n$;
to set $H_j \perp H_k$ we assume $m_{jk} = 0$ and $T_{jk} = 0$.

Problem

Describe up to a unitary equivalence irreducible systems, for which angles between H_i and H_j are in T_{ij}

Graph notation

Such systems of subspaces can be described by weighted (Coxeter) graphs. To each H_j we associate a vertex, and connect a pair of vertices j, k, $\bullet \overset{r_{jk}}{\text{———}} \bullet$ the number r_{jk} depends on the number depends on the number m_{jk} of possible angles in T_{jk} as follows:

- 2 — no edge: $\bullet \quad \bullet$, projections are orthogonal
- 3 (not written) — one angle: $\bullet\text{——}\bullet$, relations $P_iP_jP_i = \tau_{ij}P_i$, $P_jP_iP_j = \tau_{ij}P_j$
- 4 : $\bullet \overset{4}{\text{——}} \bullet$, relations $(P_iP_j)^2 = \tau_{ij}(P_iP_j)$, $(P_jP_i)^2 = \tau_{ij}(P_jP_i)$
- 5 — two angles: $\bullet \overset{5}{\text{——}} \bullet$, relations $(P_iP_jP_i - \tau_{ij}^{(1)}P_i)(P_iP_jP_i - \tau_{ij}^{(2)}P_i) = 0$, $(P_jP_iP_j - \tau_{ij}^{(1)}P_j)(P_jP_iP_j - \tau_{ij}^{(2)}P_j) = 0$

etc. Notice that for even numbers we obtain intermediate class of relations.

$*$-Algebras related to $(\Gamma, \mathbf{T})$, $\mathbf{T} = \{T_{ij}\}$

Define set f of polynomials (assume $f_{jk} = f_{kj}$)
$f_{jk}(x) = (x - \tau_{jk}^{(1)}) \dots (x - \tau_{jk}^{(m_{jk})})$, for odd weight,
$f_{jk}(x) = x(x - \tau_{jk}^{(1)}) \dots (x - \tau_{jk}^{(m_{jk})})$, for even weight.
Then the relations for the projections for odd and even weights are correspondingly
$f_{jk}(P_j P_k) P_j = f_{jk}(P_k P_j) P_k = 0$,
$f_{jk}(P_j P_k) = f_{jk}(P_k P_j) = 0$.
Such families of projections are representations of $*$-algebra

$$TL_{\Gamma,f,\perp} = \mathbb{C}\langle p_1, \dots, p_n \mid p_j^2 = p_j^* = p_j, j = 1, \dots, n \\ f_{jk}(p_j p_k) p_j^{\sigma_{jk}} = f_{jk}(p_k p_j) p_k^{\sigma_{jk}}, j \neq k \rangle$$

we call $TL_{\Gamma,f,\perp}$ the Tempreley–Lieb type algebra corresponding to Γ, f with orthogonality.

Representations of $TL_{\Gamma,f,\perp}$

Let $P_1, \dots, P_n$ be a representation of $TL_{\Gamma,f,\perp}$, and let G be the corresponding Gram matrix.
Since $\sigma(G_{jk}G_{kj}) = \sigma(G_{kj}G_{jk}) \subset T_{jk}$, $j, k = 1, \dots, n$, the condition $G \geq 0$ imposes conditions on the sets T_{jk}, $j, k = 1, \dots, n$ for a representation to exist.
For various classes and examples of graphs such conditions were studied in details.

Simple systems with orthogonality

Condition: each T_{ij} is either 0 or $\tau_{ij} < 1$

$\Gamma = (V_\Gamma, E_\Gamma)$ — simple connected graph, $V_\Gamma = \{1, \dots, n\}$,
$\tau = (\tau_{ij})_{i,j=1}^n$, $(i,j) \in E_\Gamma \iff \tau_{ij} > 0$

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Theorem (N.Popova, 2001,2002; M.Vlasenko, 2004; Yu.Samolenko, A.Strelets, 2009)

If Γ is a tree, the classification of all irreps is a $*$-finite problem.
If Γ has unique cycle, the classification of all irreps is a $*$-finite or $*$-tame problem (depends on τ).
If Γ has n cycles, $n \geq 2$, there exists τ, for which the classification of all irreps is a $*$-wild problem.

Γ is a tree

As noticed above, in the case where Γ is a tree, there can be at most one representation,
and in this case $\dim ImP_j = 1$, $j = 1, \ldots, n$. The Gram matrix is

$$G = \begin{pmatrix} 1 & & & \\ & \ddots & \sqrt{\tau_{ij}} & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

A representation exists iff $G \geq 0$, in this case G is a Gram matrix for some vectors $e_1, \ldots, e_n$, $P_j = P_{e_j}$, $j = 1, \ldots, n$.
$\dim H = n$ $(G > 0)$ or $n - 1$ $(\ker G \neq \{0\})$.

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$\dim H = n$ $(G > 0)$ or $n - 1$ $(\ker G \neq \{0\})$.
In the case $\tau_{jk} = \tau$, $j, k = 1, \dots, n$, we have $G \geq 0$ iff
$I + \sqrt{\tau} A_\Gamma \geq 0$, $\tau \leq \frac{1}{(ind\Gamma)^2}$, A_Γ — adjancency matrix of Γ.

Example: All but one collections

Consider $*$-algebra $\mathcal{P}_{abo,n}$ with generators $q, p_1, \ldots, p_n$ and relations

$$q^2 = q^* = q, \quad p_j^2 = p_j^* = p_j, \quad j = 1, \ldots, n,$$
$$p_1 + \cdots + p_n = e.$$

[N.Vasilevski, 1998]
A representation of this algebra is a family of projections $P_0, P_1, \ldots, P_n$ with $P_1 + \cdots + P_n = I$. For $n \geq 2$ the problem of unitary description of all representations is $*$-wild.

Case of single cycle

If Γ has single cycle, for any irreducible representation of $TL_{\Gamma,\tau,\perp}$ we again have that

$$\dim \operatorname{Im} P_j = 1, \quad j = 1, \ldots, n.$$

The correspondig Gram matrix can be reduced to the form

$$G_{\Gamma,\phi} = \begin{pmatrix} 1 & & & e^{i\phi}\sqrt{\tau_{1n}} \\ & \ddots & \sqrt{\tau_{jk}} & \\ & & \ddots & \\ & & & 1 \end{pmatrix}, \quad \phi \in [0, 2\pi)$$

$\dim H \in \{n, n-1, n-2\}$

The condition $G \geq 0$ can imply further restrictions on ϕ: explicit examples show that for some $\{\tau_{jk}\}$ ϕ can be arbitrary value in $[0, 2\pi)$, for other the matrix is nonnegative as ϕ is in some segment of a circle, single point or even $\emptyset$

Systems related to Coxeter graphs

Theorem (N.Popova, Yu.Samoilenko. A.Strelets, 2008)

- *If* Γ *is a tree, at most one edge* (j,k) *has* $r_{jk} > 3$*, then the description of irreps of* $TL_{\Gamma,f,\perp}$ *is* $*$*-finite.*
- *If* Γ *is a tree with two edges*

  $\ldots \bullet\overset{4}{—}\bullet \ldots \bullet\overset{4}{—}\bullet \ldots$

  $\ldots \bullet\overset{4}{—}\bullet \ldots \bullet\overset{5}{—}\bullet \ldots$

  $\ldots \bullet\overset{5}{—}\bullet \ldots \bullet\overset{5}{—}\bullet \ldots,$

  then the description of irreps of $TL_{\Gamma,f,\perp}$ *is* $*$*-tame.*
- *In the rest cases there exist such collections of angles that the description of irreps of* $TL_{\Gamma,f,\perp}$ *is* $*$*-wild.*

All but one collections

The problem of description of collections of projections $P_0, P_1, \dots, P_n$ with $P_1 + \dots + P_n = I$ and

$$\prod_{k=1}^{m_j} (P_j P_0 - \tau_j^{(k)}) P_j = \prod_{k=1}^{m_j} (P_0 P_j - \tau_j^{(k)}) P_0 = 0,$$

$j = 1, \dots, n$, is equivalent to the description of collections

$$A_j = A_j^*, \quad \sigma(A_j) \subset \{\tau_j^{(1)}, \dots, \tau_j^{(m_j)}\}, \quad j = 1, \dots, n$$

with

$$\sum_{j=1}^{n} A_j = I$$

Families of operators

Object

Families $A_1, \ldots, A_n$,

$$\sum_{k=1}^{n} A_k = \gamma I,$$

$A_k = A_k^*$, $\sigma(A_k) \subset M_k$, $k = 1, \ldots, n$.

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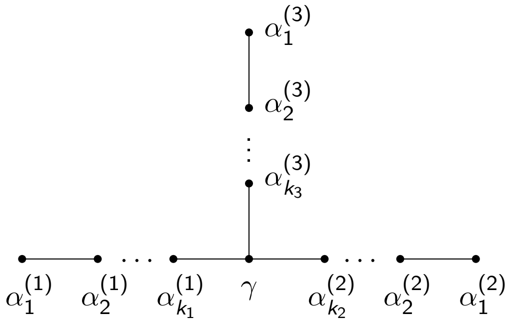
Problems

- Sets of parameters, for which a solution exists
- Structure of the operators $A_1, \ldots, A_n$

Star-shaped graphs and weights

Let Γ be a star-shaped graph. A *weight* on the graph:

$$\chi = (\alpha_1^{(1)}, \dots, \alpha_{k_1}^{(1)}; \dots ; \alpha_1^{(n)}, \dots, \alpha_{k_n}^{(n)}; \gamma),$$



Algebra related to a star-shaped graph and a weight

- $*$-Algebra $\mathcal{A}_{\Gamma,\chi}$ is generated by self-adjoint elements a_l, $l = 1, \dots, n$, which satisfy relations

$$p_l(a_l) = 0, \quad \sum_{l=1}^{n} a_l = \gamma e,$$

where $p_l(x) = x(x - a_1^{(l)}) \dots (x - a_{k_l}^{(l)})$, $l = 1, \dots, n$.

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where $p_l(x) = x(x - a_1^{(l)}) \dots (x - a_{k_l}^{(l)})$, $l = 1, \dots, n$.

- $*$-Representations of this algebra are n-tuples $A_1, \dots, A_n$ with $A_1 + \dots + A_n = \gamma I$ and the spectrum of each A_l is contained in $\{0, a_1^{(l)}, \dots, a_{k_l}^{(l)}\} = M_l$.

Algebra related to a star-shaped graph and a weight

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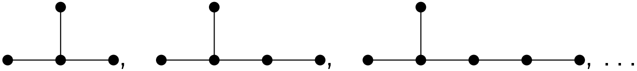
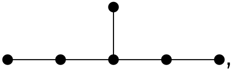
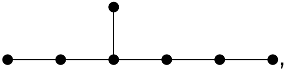
- *Problems*:
  — For which χ there exist $*$-representations of $\mathcal{A}_{\Gamma,\chi}$?
  — What is the structure of $*$-representations?

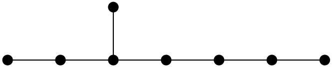
Star-shaped Dynkin graphs

A_n, $n \geq 1$: 

D_n, $n \geq 4$: 

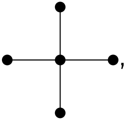
E_6: 

E_7: 

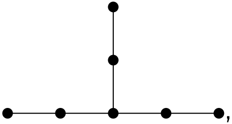
E_8: 

Star-shaped extended Dynkin graphs

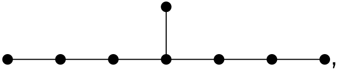
$\tilde{D}_4$:



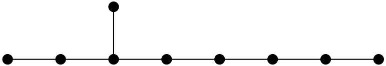
$\tilde{E}_6$:



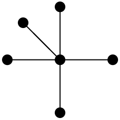
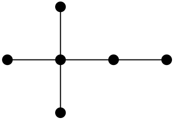
$\tilde{E}_7$:

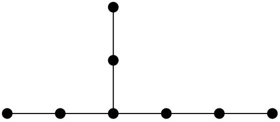


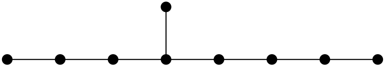
$\tilde{E}_8$:

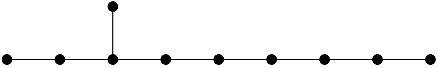


Star-shaped critical graphs

$\Gamma_{(1,1,1,1,1)}$: , $\Gamma_{(1,1,1,2)}$: ,

$\Gamma_{(2,2,3)}$: ,

$\Gamma_{(1,3,4)}$: ,

$\Gamma_{(1,2,6)}$: 

Any graph which properly contains an extended Dynkin graph, contains one of the listed above graphs

Irreducible representations

Theorem

If Γ is a Dynkin graph, the correponding algebra $\mathcal{A}_{\Gamma,\chi}$ is finite-dimensional for any χ, it has finite number of irreducible $*$-representations, and they are finite-dimensional.

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Theorem (S.Albeverio, V.O., Yu.Samoilenko. (2007))

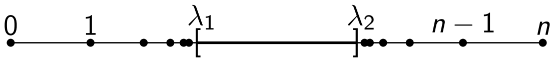
Let Γ contains an extended Dynkin graph properly. There exists a weight χ such that $\mathcal{A}_{\Gamma,\chi}$ has infinite-dimensional irreducible $*$-representation.

Example

Sums of projections $P_1 + \cdots + P_n = \gamma I$.
$\Sigma_n = \{\gamma \in \mathbb{R} \mid \exists\ P_1 + \cdots + P_n = \gamma I\}$

Theorem (S.Kruglyak, V.Rabanovich, Yu.Samolienko, 2002)

$\Sigma_n = \Lambda_n \cup [\frac{1}{2}(n - \sqrt{n^2 - 4n}), \frac{1}{2}(n + \sqrt{n^2 - 4n})] \cup n - \Lambda$, *where*
$\Lambda_n = \{\frac{1}{2}\operatorname{cth}(k \operatorname{arcth}(\frac{1}{2}\sqrt{n}))(n - \sqrt{n^2 - 4n}), k \in \mathbb{N}\}$



$$\lambda_{1,2} = \frac{n \pm \sqrt{n^2 - 4n}}{2}$$

Four-tuples of projections with scalar sum

Let P_1, P_2, P_3, P_4 be projections satisfying

$$P_1 + P_2 + P_3 + P_4 = \lambda I, \quad \lambda \in \mathbb{R}.$$

Theorem (V.O., Yu.Samoilenko, 1998)

- *A solution exists for* $\lambda \in \{2 \pm \frac{1}{k+s} \mid s \in \{1/2, 1\}, k = 0, 1, \dots\} \cup \{2\}$;
- *For* $\lambda = 2 \pm \frac{2}{2k+1}$, $k \geq 0$, *there exists one irrep of dimension* $2k + 1$;
- *For* $\lambda = 2 \pm \frac{1}{k+1}$, $k \geq 0$, *there exists 4 irreps of dimension* $k + 1$;
- *For* $\lambda = 2$ *there exist two-parametric family of irreps of dimension 2 and 6 irreps of dimension 1.*

Sets of parameters, for which there exist representations. Example: Families related to $\tilde{E}_6$

Theorem

$A_1 + A_2 + A_3 = \gamma I$, $\sigma(A_k) \subset \{0, 1, 2\}$ *have solutions iff*

$$\gamma \in W_{\tilde{E}_6} = \Big\{3 \pm \frac{1}{k+s} \mid k = 0, 1, \ldots; s \in \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\}\Big\} \cup \{3\}$$

Similar theorems hold for all extended Dynkin graphs.