A NOTE ON WEIGHTED ORLICZ ALGEBRAS

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Young Function

Definition (Young Function)[H. Hudzik, 1985]

A non-zero function $\Phi : \mathbb{R} \to [0, +\infty]$ is called a Young function if

(i) Φ is convex,

(ii) Φ is even,

(iii) $\Phi(0) = 0$.

Note

Note that this definition of Young functions allows them to take the value ∞, and hence they may be discontinuous at the point where they take the value infinity. However, unless otherwise specified we will consider only real-valued Young functions. Such a Φ is necesserily continuous, and tends to infinity as x tends to infinity.

Complementary Function

Definition (Complementary Function)

Given a Young function Φ, the complementary function Ψ of Φ is given by

$$\Psi(y) = \sup\{x|y| - \Phi(x) \mid x \geq 0\}$$

for $y \in \mathbb{R}$. If Ψ is the complementary function of Φ, then Φ is the complementary function of Ψ and (Φ, Ψ) is called a complementary pair of Young functions.

Note

Even if Φ is finite valued it may happen that Ψ takes infinite values.

Example

1) Let $1 < p < +\infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $\Phi(x) = \frac{|x|^p}{p}$, $x \in \mathbb{R}$, and $\Psi(x) = \frac{|x|^q}{q}$, $x \in \mathbb{R}$, are a complementary pair of Young functions.

Example

2) In particular, when $p = 1$, the complementary function of $\Phi(x) = |x|$ is

$$\Psi(x) = \begin{cases} 0, & 0 \leq |x| \leq 1, \\ +\infty, & |x| > 1. \end{cases}$$

Weighted Orlicz Spaces

Definition (Weighted Orlicz Space)

Let G be a locally compact group with left Haar measure μ and w be a weight on G (i.e., w is a positive, Borel measurable function such that $w(xy) \leq w(x)w(y)$ for all $x, y \in G$). Given a Young function Φ, the weighted Orlicz space $L^{\Phi}_{w}(G)$ is defined by

$$L^{\Phi}_{w}(G) := \left\{ f : G \to K \mid \exists \alpha > 0, \int_G \Phi(\alpha |fw|) d\mu < +\infty \right\}.$$

Then $L^{\Phi}_{w}(G)$ becomes a Banach space under the norm $|| \cdot ||_{\Phi,w}$ (called the weighted Orlicz norm) defined for $f \in L^{\Phi}_{w}(G)$ by

$$||f||_{\Phi,w} := \sup \left\{ \int_G |fwv| d\mu \mid v \in L^{\Psi}(G), \int_G \Psi(|v|) d\mu \leq 1 \right\},$$

where Ψ is the complementary function of Φ.

For $f \in L^{\Phi}_{w}(G)$, one can also define the norm

$$||f||^{\circ}_{\Phi,w} = \inf\left\{k > 0 \mid \int_G \Phi\left(\frac{|fw|}{k}\right) d\mu \leq 1\right\},$$

which is called the weighted Luxemburg norm and is equivalent to the weighted Orlicz norm.

Recall...

Notice that if $\Phi(x) = \frac{|x|^p}{p}$, $1 \leq p < +\infty$, then $L^{\Phi}_{w}(G)$ becomes the classical weighted Lebesgue space $L^p(G)$.

If

$$\Psi(x) = \begin{cases} 0, & 0 \leq |x| \leq 1, \\ +\infty, & |x| > 1, \end{cases}$$

then $L^{\Psi}_{w}(G) = L^{\infty}_{w}(G)$.

Definition (Δ_2 Condition)

Let Φ be a Young function. We say that Φ satisfies the Δ_2 condition whenever there exists a $K > 0$ such that

$$\Phi(2x) \leq K\Phi(x)$$

for all $x \geq 0$, and we write $\Phi \in \Delta_2$ in such a case.

Mostly we consider the Δ_2 condition for the Young function Φ.

Examples

- For $1 \leq p < \infty$, if $\Phi(x) = \frac{|x|^p}{p}$, $x \in \mathbb{R}$, then $\Phi \in \Delta_2$.
- If $\Phi(x) = e^{|x|-1}$, $x \in \mathbb{R}$, then $\Phi \notin \Delta_2$.

Dual Space of $L^{\Phi}_{w}(G)$

Theorem (Dual Space)

Let G be a locally compact group and w be a weight on G. If Φ be a Young function such that $\Phi \in \Delta_2$ and Ψ is the complementary function of Φ, then the dual space of $(L^{\Phi}_{w}(G), ||\cdot||_{\Phi,w})$ is $L^{\Psi}_{w^{-1}}(G)$ formed by all measurable functions g on G such that $\frac{g}{w} \in L^{\Psi}(G)$ and endowed with the norm $||\cdot||^{\circ}_{\Psi,w^{-1}}$ defined for $g \in L^{\Psi}_{w^{-1}}(G)$ by

$$||g||^{\circ}_{\Psi,w^{-1}} := \left\|\frac{g}{w}\right\|^{\circ}_{\Psi} = \inf\left\{k > 0 : \int_G \Psi\left(\frac{g}{kw}\right) d\mu \leq 1\right\}.$$

Corollary

Let (Φ, Ψ) be a complementary pair of Young functions such that $\Phi, \Psi \in \Delta_2$. Then the weighted Orlicz space $L^{\Phi}_{w}(G)$ is reflexive.

Basic Properties of $L^{\Phi}_w(G)$

Proposition

Let Φ be a Young function such that $\Phi \in \Delta_2$ and $f \in L^{\Phi}_w(G)$. Then

i) $\overline{C_c(G)}^{||\cdot||_{\Phi,w}} = L^{\Phi}_w(G)$,

ii) for every $x \in G$, $L_x f \in L^{\Phi}_w(G)$ and $||L_x f||_{\Phi,w} \leq w(x)||f||_{\Phi,w}$,

iii) the map

$$\begin{array}{ccc} G & \to & L^{\Phi}_w(G) \\ x & \mapsto & L_x f \end{array}$$

is continuous.

Weighted Orlicz Algebra with Respect to Convolution Multiplication

Theorem [H. Hudzik, 1985]

For G is a locally compact abelian group,
$L^{\Phi}(G)$ is a Banach algebra w.r.t. convolution $\Leftrightarrow L^{\Phi}(G) \subseteq L^{1}(G)$.

Theorem (Weighted Orlicz Algebra)

Let G be a locally compact group, w be a weight on G and let Φ be a Young function. If $L^{\Phi}_{w}(G) \subseteq L^{1}_{w}(G)$, then the weighted Orlicz space $(L^{\Phi}_{w}(G), ||\cdot||_{\Phi,w})$ is a Banach algebra w.r.t. convolution, which we call the weighted Orlicz algebra.

Note that the converse is not true in general. For $\Phi(x) = \frac{|x|^p}{p}$, $p > 1$, $L^{p}_{w}(G)$ is a Banach algebra (Kuznetsova, 2006), but it is not in $L^{1}_{w}(G)$.

Observation

For Φ is a Young function with $\Phi'_+(0) > 0$, then the inclusion $L^\Phi_w(G) \subseteq L^1_w(G)$ is true. So $L^\Phi_w(G)$ becomes a weighted Orlicz algebra. In particular, if G is non compact and abelian locally compact group, then $\Phi'_+(0) > 0 \iff L^\Phi_w(G) \subseteq L^1_w(G)$ (Hudzik, 1985).

Observation

Without any assumption on the Young function Φ, we can have the weighted Orlicz space $L^\Phi_w(G)$ as a left Banach $L^1_w(G)$-module w.r.t. convolution.

Henceforth, we assume that a Young function Φ satisfies the Δ_2 condition.

Theorem

The weighted Orlicz algebra $L^{\Phi}_{w}(G)$ has a left approximate identity consisting of compactly supported functions that are bounded w.r.t. the $||\cdot||_{1,w}$ norm.

Theorem

The weighted Orlicz algebra $L^{\Phi}_{w}(G)$ has an identity if and only if G is discrete.

Theorem

Let the complementary function of Φ satisfy the Δ_2 condition. If G is non-discrete, then the weighted Orlicz algebra $L^{\Phi}_{w}(G)$ admits no bounded left approximate identity w.r.t. the $||\cdot||_{\Phi,w}$ norm.

Let $L^\Phi_w(G)$ be a weighted Orlicz algebra. The closed left ideals of $L^\Phi_w(G)$ turn out to be nothing but the closed left translation-invariant subspaces of $L^\Phi_w(G)$.

Theorem

Let $L^\Phi_w(G)$ be a weighted Orlicz algebra and let I be a closed linear subspace of $L^\Phi_w(G)$. Then

$$I \text{ is a left ideal} \quad \Leftrightarrow \quad \forall x \in G, L_x(I) \subseteq I.$$

Observation

If $w = 1$, then the closed left ideals of the Orlicz algebra $L^\Phi(G)$ coincide with the closed left translation-invariant subspaces.

Proposition

Let Φ be a Young function such that $\Phi'_+(0) > 0$. Then the weighted Orlicz algebra $L^\Phi_w(G)$ is a left ideal in $L^1_w(G)$.

Let G be a locally compact abelian group, w be a weight and let Φ be a Young function. We now describe the maximal ideal space (spectrum) $\Delta(L_w^\Phi(G))$ of the commutative weighted Orlicz algebra $L_w^\Phi(G)$ in terms of the so-called generalized characters of G determined by the complementary function Ψ of Φ and a weight w.

Note

If G is abelian, then the weighted Orlicz algebra $L_w^\Phi(G)$ is a commutative.

Definition

Let G be a locally compact abelian group, w be a weight and Φ be a Young function with the complementary function Ψ. A generalized character determined by the function Ψ and a weight w on G is a continuous function $\gamma : G \to \mathbb{C}\backslash\{0\}$ satisfying the conditions

i) $\gamma(x+y) = \gamma(x)\gamma(y)$ for all $x, y \in G$,

ii) $\frac{\gamma}{w} \in L^\Psi(G)$.

Let $\widehat{G_\Psi}(w)$ denote the set of all generalized characters of G equipped with the topology of uniform convergence on compact subsets of G.

Theorem

Let G be a locally compact abelian group and let $L^{\Phi}_w(G)$ be a weighted Orlicz algebra. For $\gamma \in \widehat{G_{\Psi}}(w)$, define $\varphi_\gamma : L^{\Phi}_w(G) \to \mathbb{C}$ by

$$\varphi_\gamma(f) = \int_G f(x)\overline{\gamma(x)}d\mu(x)\ ,\ f \in L^{\Phi}_w(G).$$

Then $\varphi_\gamma \in \Delta(L^{\Phi}_w(G))$, and the map $\gamma \mapsto \varphi_\gamma$ is a bijection between $\widehat{G_{\Psi}}(w)$ and $\Delta(L^{\Phi}_w(G))$.

Observation

If $w = 1$, then for the Orlicz algebra $L^{\Phi}(G)$, $\Delta(L^{\Phi}(G)) \cong \widehat{G_{\Psi}}$.

Observation

Let Φ be a Young function with $\Phi'_+(0) > 0$ and w be a weight such that $w(x) \geq 1$ for all $x \in G$. Then $L^{\Phi}_w(G)$ and $L^{\Phi}(G)$ become commutative Banach algebras and $L^{\Phi}_w(G) \subseteq L^{\Phi}(G)$ is true. Hence we have $\widehat{G_{\Psi}} \subseteq \widehat{G_{\Psi}}(w)$.

Proposition

The weighted Orlicz algebra $L^{\Phi}_{w}(G)$ is not radical.

Theorem

The weighted Orlicz algebra $L^{\Phi}_{w}(G)$ is semi-simple.

Sketch of Proof

Since $L^{\Phi}_{w}(G)$ is not a radical algebra, there exists a $\varphi \in \Delta(L^{\Phi}_{w}(G))$ and this φ is determined by $\gamma \in \widehat{G_{\Psi}}(w)$ uniquely. For each $\alpha \in \hat{G}$, define $\varphi_{\alpha} \in (L^{\Phi}_{w}(G))^{*}$ by

$$\varphi_{\alpha}(f) = \int_{G} f\overline{\alpha\gamma}d\mu.$$

Then $\varphi_{\alpha} \in \Delta(L^{\Phi}_{w}(G))$ since $\alpha\gamma \in \widehat{G_{\Psi}}(w)$ for all $\alpha \in \hat{G}$. Let f be an element of the radical of $L^{\Phi}_{w}(G)$. Then $\widehat{f\overline{\gamma}} \in L^{1}(G)$ and

$$\widehat{f\overline{\gamma}}(\alpha) = \varphi_{\alpha}(f) = 0$$

for all $\alpha \in \hat{G}$. So we get $f = 0$.

Observation

If $w = 1$, then the Orlicz algebra $L^{\Phi}(G)$ is not radical, but is semi-simple.

Note

If $L^{\Phi}_{w}(G) \subseteq L^{1}_{w}(G)$, then we have seen above that $L^{\Phi}_{w}(G)$ is a Banach algebra w.r.t. convolution, but the converse is not true in general. However, the following theorem shows that if $L^{\Phi}_{w}(G)$ is a Banach algebra w.r.t. convolution, then one can always assume that the inclusion $L^{\Phi}_{w}(G) \subseteq L^{1}(G)$ is true, similar to the case $L^{p}_{w}(G)$ (Kuznetsova, 2006).

Lemma

Let w be a weight on G and let Φ be a Young function such that $\Phi \in \Delta_2$ with the complementary function Ψ. Then the following are equivalent:

i) $\frac{1}{w} \in L^{\Psi}(G)$,

ii) $L^{\Phi}_{w}(G) \subseteq L^{1}(G)$,

iii) $L^{\infty}(G) \subseteq L^{\Psi}_{w^{-1}}(G)$.

Theorem

Each weighted Orlicz algebra $L^{\Phi}_{w}(G)$ is isometrically isomorphic to an algebra $L^{\Phi}_{\tilde{w}}(G)$ with a weight $\tilde{w}$ satisfying $\frac{1}{\tilde{w}} \in L^{\Psi}(G)$.

Weighted Orlicz Algebra with Respect to Pointwise Multiplication

H. Hudzik (1985) gives necessary and sufficient conditions for an Orlicz space to be a Banach algebra with respect to pointwise multiplication on the measure space (X, Σ, μ). We adapt the results of H. Hudzik to a locally compact group G.

Proposition

Let G be a locally compact group and w be a weight on G. If Φ is a strictly increasing Young function, then the following statements are equivalent for $\lim_{x\to\infty} \frac{\Phi(x)}{x} = +\infty$:

i) $L^{\Phi}_w(G) \subseteq L^{\infty}_w(G)$,

ii) G is discrete,

iii) $L^1_w(G) \subseteq L^{\Phi}_w(G)$.

We need the limit condition for $iii) \Rightarrow ii)$.

Corollary

If $G = \mathbb{Z}$, then the weighted Orlicz sequence spaces denoted by $L^{\Phi}_{w}(\mathbb{Z}) = \ell^{\Phi}_{w}$ satisfy

$$\ell^{1}_{w} \subseteq \ell^{\Phi}_{w} \subseteq \ell^{\infty}_{w}.$$

Theorem

Let G be a locally compact group and w a weight on G . If Φ is a strictly increasing Young function, then $L^{\Phi}_{w}(G)$ is a Banach algebra w.r.t. pointwise multiplication if and only if $L^{\Phi}_{w}(G) \subseteq L^{\infty}_{w}(G)$.

Observation

Under the same conditions as in the previous theorem, the weighted Orlicz space $L^{\Phi}_{w}(G)$ is a Banach algebra w.r.t. pointwise multiplication if and only if G is discrete.