Quantum polydisk, quantum ball, and a q-analog of Poincaré's theorem

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Poincaré's Theorem

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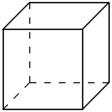

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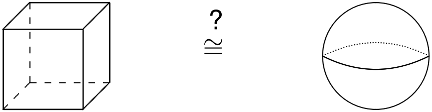

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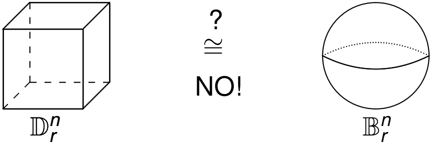

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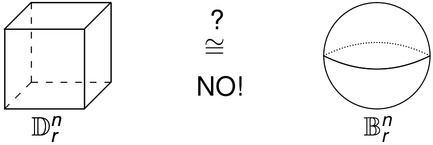

Theorem (H. Poincaré, 1907)

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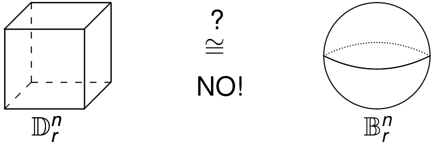

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- **Goal:** prove a noncommutative analog of Poincaré's theorem.

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(S2) for each compact set $K \subset X$, the holomorphically convex hull

$$\widehat{K} = \left\{ x \in X : |f(x)| \le \sup_{y \in K} |f(y)| \quad \forall f \in \mathscr{O}(X) \right\}$$

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Theorem (H. Cartan, P. Thullen, 1932)

A domain $D \subset \mathbb{C}^n$ is a Stein manifold iff D is a domain of holomorphy (i.e., there exists $f \in \mathscr{O}(D)$ which cannot be holomorphically extended to a larger domain).

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The functor

$$\Big\{\text{Stein manifolds}\Big\} \to \Big\{\text{Fréchet algebras}\Big\},$$
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- As a consequence, $X \cong Y \iff \mathscr{O}(X) \cong \mathscr{O}(Y)$.

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- Inetrpret them as "deformations" of $\mathscr{O}(\mathbb{D}_r^n)$ and $\mathscr{O}(\mathbb{B}_r^n)$, respectively;
- Study whether or not $\mathscr{O}_q(\mathbb{D}_r^n)$ and $\mathscr{O}_q(\mathbb{B}_r^n)$ are topologically isomorphic.

Algebraic prototype: Quantum affine space

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Both $\mathscr{O}_q(\mathbb{D}_r^n)$ and $\mathscr{O}_q(\mathbb{B}_r^n)$ will be completions of $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$.

Arens-Michael envelopes

Definitions

- Let A be a $\mathbb{C}$-algebra (associative, with 1). A seminorm $\|\cdot\|$ on A is submultiplicative if $\|ab\| \le \|a\|\|b\|$ for all $a, b \in A$.

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Examples

All Banach algebras, $C^\infty(M)$, $\mathscr{O}(X)$ etc.

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2. (P., 2004). Let $(X, \mathscr{O}_X^{\mathrm{reg}})$ be an affine scheme of finite type, and let $(X_h, \mathscr{O}_{X_h})$ be the associated Stein space. Then $\mathscr{O}^{\mathrm{reg}}(X)\widehat{\ } = \mathscr{O}(X_h)$.

Holomorphic functions on the quantum affine space

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Explicit construction: define

$$w_q \colon \mathbb{Z}_+^n \to \mathbb{R}_+, \quad w_q(k) = \begin{cases} 1 & \text{if } |q| \geq 1, \\ |q|^{\sum_{i<j} k_i k_j} & \text{if } |q| < 1. \end{cases}$$

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- *Each norm* $\| \cdot \|_\rho$ *is submultiplicative.*

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The algebra of holomorphic functions on the quantum n-polydisk of radius $r \in (0, +\infty]$ is

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Theorem (Aizenberg, Mityagin, 1967)

$$\mathscr{O}(D) \cong \Big\{ f = \sum_{k \in \mathbb{Z}_+^n} c_k z^k : \|f\|_s = \sum_{k \in \mathbb{Z}_+^n} |c_k| b_k(D) s^{|k|} < \infty \ \forall s \in (0,1) \Big\}.$$

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- $\mathrm{Pol}_q(\mathbb{C}^n)$ is one of the basic examples in the theory of "quantum bounded symmetric domains"
(L. Vaksman, O. Bershtein, Y. Kolisnyk, D. Proskurin, S. Shklyarov, S. Sinel'shchikov, A. Stolin, L. Turowska. . . ; 1998–present time).

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Theorem (Pusz, Woronowicz, 1989)

There exists a faithful irreducible $*$-representation $\pi\colon \operatorname{Pol}_q(\mathbb{C}^n) \to \mathscr{B}(H)$ uniquely determined by

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- For $n = 1$: S. Klimek and A. Lesniewski (1993).

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- If $q = 1$, then the completion of $\mathbb{C}[z_1, \dots, z_n] = \mathscr{O}^{\mathrm{reg}}(\mathbb{C}^n)$ w.r.t. the family $\{\|\cdot\|_\rho^\infty : 0 < \rho < r\}$ of norms is topologically isomorphic to $\mathscr{O}(\mathbb{B}_r^n)$.

A relation to Vaksman's quantum ball

- The subalgebra of $\mathrm{Pol}_q(\mathbb{C}^n)$ generated by $x_1, \ldots, x_n$ is $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$.
- For each $\rho > 0$ consider $\gamma_\rho \in \mathrm{Aut}(\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n))$ given by $\gamma_\rho(x_i) = \rho x_i$ $(i = 1, \ldots, n)$.
- Define a norm on $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ by $\|a\|_\rho^\infty = \|\gamma_\rho(a)\|^\infty$.
- The above norm is a q-analog of $\|a\|_\rho^\infty = \sup_{z \in \bar{\mathbb{B}}_\rho^n} |a(z)|$.
- If $q = 1$, then the completion of $\mathbb{C}[z_1, \ldots, z_n] = \mathscr{O}^{\mathrm{reg}}(\mathbb{C}^n)$ w.r.t. the family $\{\|\cdot\|_\rho^\infty : 0 < \rho < r\}$ of norms is topologically isomorphic to $\mathscr{O}(\mathbb{B}_r^n)$.

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Holomorphic functions on the free polydisk

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Definition (J. L. Taylor, 1972)

The algebra of holomorphic functions on the free n-polydisk of radius $r \in (0, +\infty]$ is

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- If $r = \infty$, then $\mathscr{F}(\mathbb{D}_r^n) = \mathscr{F}(\mathbb{C}^n)$ is the Arens-Michael envelope of F_n.

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Theorem

$$\mathscr{O}_q(\mathbb{D}_r^n) \cong \mathscr{F}(\mathbb{D}_r^n)/\overline{(\zeta_i\zeta_j - q\zeta_j\zeta_i,\ i < j)}.$$

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Moreover, for each $\rho \in (0,r)$ the norm

$$\|a\|_\rho = \sum_{k\in\mathbb{Z}_+^n} |c_k| w_q(k)\rho^{|k|} \qquad \Big(a = \sum_{k\in\mathbb{Z}_+^n} c_k x^k\Big)$$

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Definition

$\mathscr{F}(\mathbb{B}_r^n)$ is the algebra of holomorphic functions on the free n-ball of radius $r \in (0, +\infty]$.

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Moreover, for each $\rho \in (0, r)$ the norm

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$$\begin{aligned} A \times_X A \to A, &\quad (a, b) \mapsto a + b, \\ \mathbb{C} \times A \to A, &\quad (\lambda, a) \mapsto \lambda a, \\ A \times_X A \to A, &\quad (a, b) \mapsto ab, \end{aligned}$$

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  are continuous;
- There exists a family $\{\|\cdot\|_i : i \in I\}$ of seminorms on A making each A_x into a Fréchet algebra and such that, for each $x \in X$, the "rectangles"

$$\Big\{a \in A : p(a) \in U,\ \|a\|_i < \varepsilon\Big\} \qquad (i \in I,\ \varepsilon > 0,\ U \ni x \text{ open })$$

  form a neighborhood base of $0 \in A_x$.

Deformations of the polydisk and of the ball

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Let $r \in (0, +\infty]$.

- *There exists a Fréchet algebra bundle* (D, p) *over* $\mathbb{C}^\times$ *such that for each* $q \in \mathbb{C}^\times$ *we have* $D_q \cong \mathscr{O}_q(\mathbb{D}_r^n)$.

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(i) *If* $n \geq 2$, $r < \infty$, *and* $|q| = 1$, *then* $\mathscr{O}_q(\mathbb{D}_r^n)$ *and* $\mathscr{O}_q(\mathbb{B}_r^n)$ *are not topologically isomorphic.*

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A q-analog of Poincaré's theorem

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(i) *If $n \geq 2$, $r < \infty$, and $|q| = 1$, then $\mathscr{O}_q(\mathbb{D}_r^n)$ and $\mathscr{O}_q(\mathbb{B}_r^n)$ are not topologically isomorphic.*

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- Hence $n = 1$. $\square$

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Problem

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