Derivations of the non-commutative Schwartz space

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Plan

1. Introduction

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2. Automatic continuity of positive functionals

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3. Automatic continuity of derivations

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$$s = \left\{ \xi = (\xi_j)_{j\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}} \colon\ |\xi|_k^2 := \sum_{j=1}^{+\infty} |\xi_j|^2 j^{2k} < +\infty \text{ for all } k \in \mathbb{N}_0 \right\},$$

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$$L(s',s):=\{\text{linear and continuous maps } x\colon\ s'\to s\},$$

topology on $L(s',s)$ given by $\|x\|_k:=\sup\{|x\xi|_k\colon\ |\xi|_k'\leqslant 1\}$.

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$$\langle \xi, \eta \rangle := \sum_{j=1}^{+\infty} \xi_j \overline{\eta}_j \quad \xi \in s,\ \eta \in s' \quad \text{we have}$$

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$$\langle\xi,\eta\rangle:=\sum_{j=1}^{+\infty}\xi_j\overline{\eta}_j\quad \xi\in s,\ \eta\in s' \quad \text{we have}$$

$$\langle x^*\xi,\eta\rangle:=\langle\xi,x\eta\rangle,\quad \xi,\eta\in s',\ x\in L(s',s).$$

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Definition

$$\mathcal{S} := (L(s', s), \cdot, *).$$

Non-commutative Schwartz space because

$$\mathcal{S} = s\tilde{\otimes}_{\varepsilon}s \simeq s \simeq \mathcal{S}(\mathbb{R}).$$

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Algebra of smooth operators because

$$\mathcal{S} \simeq C^\infty([a, b]).$$

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Proposition (Domański, 2012)

If $\mathcal{S}_1 := \mathcal{S} \oplus \mathbb{C}$ *then* $\sigma_{\mathcal{S}_1}(x) = \sigma_{\mathcal{B}(\ell_2)}(x)$.

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Approximate identities

Definition

$(u_\alpha)_{\alpha\in\Lambda}$ is an a.i. if $u_\alpha x\to x$ and $xu_\alpha\to x$ for all x. It is bounded if the set of u_α's is bounded. It is sequential if $\Lambda=\mathbb{N}$.

Proposition

If $u_n := \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$ *then* $(u_n)_n$ *is a sequential a.i.*

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$\|x\|_{k,p} := \|(x_{ij}(ij)^k)_{i,j\in\mathbb{N}}\|_{\ell_p}$ gives the topology of $\mathcal{S}$ $(1\leqslant p\leqslant +\infty)$.

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$\|u_n x - x\|_{k,p},\ \|x u_n - x\|_{k,p} \to 0 \ \ \forall\, k \in \mathbb{N}$.

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$\|u_n x - x\|_{k,p}, \|x u_n - x\|_{k,p} \to 0 \ \ \forall\, k \in \mathbb{N}$.
If $(u_\alpha)_\alpha$ is a b.a.i. then $u_\alpha \to u =:$ unit in $\mathcal{S}$. But $\mathcal{S} \subset \mathcal{K}(\ell_2)$. □

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Key ingredient: functional calculus of Ciaś.

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Definition

A linear $\phi\colon \mathcal{S} \to \mathbb{C}$ is *positive* ($\phi \geqslant 0$) if $\phi(\mathcal{S}_+) \subset [0, +\infty)$.

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Theorem (Dixon, 1981; Namioka, 1957)

Let A be a Fréchet algebra with involution. If A^2 is closed, $\operatorname{codim}_A A^2 < +\infty$ and A_+ is closed then every positive functional on A is continuous.

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Reminder.

$$A^2 := \Big\{ \sum_{j=1}^{n} x_j y_j : \; x_j, y_j \in A, \, n \in \mathbb{N} \Big\}.$$

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Question

Suppose $B \subset \mathcal{S}_{sa}$ is bounded. What about $B_+ := \{x_+ : \ x \in B\}$?

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Lemma

If $A = (a_{ij})_{i,j \in \mathbb{N}} \in \mathcal{S}_+$ then $\|A\|_{k,\infty} = \sup\{a_{jj} j^{2k} : \ j \in \mathbb{N}\} \ \ \forall\, k \in \mathbb{N}$.

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Proof.

Let $P, Q, T \in \mathcal{B}(H)$ with $P, Q \geqslant 0$. Then $\begin{pmatrix} P & T \\ T^* & Q \end{pmatrix} \geqslant 0$ in $\mathcal{B}(H \oplus H)$ if and only if $|\langle Tx, y\rangle|^2 \leqslant \langle Px, x\rangle\langle Qy, y\rangle \ \ \forall\, x, y \in H$.

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Decompose A into block matrices. □

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Corollaries

1. $\forall\, x \in \mathcal{S}, n \in \mathbb{N}\colon\ \|x\|_{n,\infty}^2 \leqslant \max\{\|x^*x\|_{2n,\infty}, \|xx^*\|_{2n,\infty}\}.$

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1. $\forall\, x \in \mathcal{S}, n \in \mathbb{N}\colon\ \|x\|^2_{n,\infty} \leqslant \max\{\|x^*x\|_{2n,\infty}, \|xx^*\|_{2n,\infty}\}$.
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3. *A subset $A \subset \mathcal{S}$ is bounded if and only if the sets $\{x^*x\colon\ x \in A\}$ and $\{xx^*\colon\ x \in A\}$ are both bounded.*

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Theorem

Every positive functional on $\mathcal{S}$ is continuous.

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Proof.

Let $\phi \geqslant 0$ and suppose $\sup\{\phi(x) : \ x \in B\} = +\infty$ for some bounded $B \subset \mathcal{S}_+$.

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Proposition

$\phi_\xi(x) := \langle x\xi, \xi\rangle$ *is extendable if and only if* $\xi \in \ell_2$.

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Definition

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$(E, \cdot)$ is a left A-module if $(a, x) \mapsto a \cdot x,\ A \times E \to E$ satisfies

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2. *Ideals are modules.*
3. *E is a left $L(E)$-module with $(T, x) \mapsto Tx,\ L(E) \times E \to E$.*
4. *If E is a left A-module then E^* becomes a right A-module with*

$$\phi \cdot a(x) := \phi(a \cdot x), \quad \phi \in E^*, x \in E\, a \in A.$$

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Theorem (B.E. Johnson, 1972)

TFAE for a locally compact group G:

1. *G is amenable,*

Theorem

If D is a continuous derivation on a Banach algebra then e^D is an automorphism of this algebra.

Definition

A locally compact group is *amenable* if there exists an invariant mean on $L^\infty(G)$.

All compact and abelian groups are amenable.

Theorem (B.E. Johnson, 1972)

TFAE for a locally compact group G:

1. *G is amenable,*
2. *for every $L^1(G)$-bimodule E and every continuous derivation $D\colon L^1(G) \to E'$ there is an $x \in E'$ with $D(a) = a \cdot x - x \cdot a$.*

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Definition

A (Banach, Fréchet,...) algebra A is *amenable* if for every (Banach, Fréchet,...) A-bimodule E any continuous derivation $D\colon A \to E'$ is inner.

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First Fundamental Question

When a continuous derivation is inner?

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A (Banach, Fréchet,...) algebra A is *super-amenable* if for every (Banach, Fréchet,...) A-bimodule E any continuous derivation $D\colon A \to E$ is inner.

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Let A be a super-amenable Banach algebra.

1. *If A is commutative then $A \cong \mathbb{C}^n$.*

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Let A be a super-amenable Banach algebra.

1. *If A is commutative then $A \cong \mathbb{C}^n$.*
2. *If A has the AP then $A \cong M_{n_1} \oplus \ldots \oplus M_{n_k}$.*

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Theorem ($\Rightarrow$ Connes, 1978; $\Leftarrow$ Haagerup, 1983)

A C^*-algebra is amenable if and only if it is nuclear.

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Theorem (Pirkovskii, 2009)

The non-commutative Schwartz space is not amenable.

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Second Fundamental Question

When a derivation is automatically continuous?

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Second Fundamental Question

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Theorem (Ringrose, 1971)

Every derivation from a C^*-algebra into any of its bimodules is automatically continuous.

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Theorem

Every derivation from the non-commutative Schwartz space into any of its bimodules is automatically continuous.

Proof.

Let $\delta\colon \mathcal{S} \to E$ be given.
Enough to take Banach bimodules (due to Pirkovskii, 2008).

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J is a closed two-sided ideal and $\delta\,|_J$ is continuous.

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J is a closed two-sided ideal and $\delta\mid_J$ is continuous.
$\operatorname{codim}_{\mathcal{S}} J < +\infty$. If not, $x \mapsto x\delta(a)$ is discontinuous for a suitably chosen $a \in \mathcal{S}$. □