Computation of analytic capacity and application to the subadditivity problem

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Joint work with

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Notation for the whole talk:

- $K :=$ a compact subset of $\mathbb{C}$
- $\Omega :=$ the unbounded component of $\mathbb{C}_\infty \setminus K$

Analytic capacity of K

$$\gamma(K) := \max\Big\{|f'(\infty)| : f \in H^\infty(\Omega),\ \|f\|_\infty \le 1\Big\}.$$

- $f'(\infty) := a_1$, where $f(z) = \sum_{j\ge0} a_j z^{-j}$ near ∞.

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- $f'(\infty) := a_1$, where $f(z) = \sum_{j\geq 0} a_j z^{-j}$ near ∞.
- If the max > 0, it is attained by a unique f (Ahlfors function). Automatically $f(\infty) = 0$.

Some facts about analytic capacity

History

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Examples

- $\gamma(\text{disk}) = \text{radius}$
- If $K \subset \mathbb{R}$ then $\gamma(K) = |K|/4$
- $m_1(K) = 0 \;\Rightarrow\; \gamma(K) = 0 \;\Rightarrow\; m_{1+\epsilon}(K) = 0$

Upper and lower bounds for analytic capacity

Assume K bounded by N disjoint piecewise-analytic Jordan curves. Write $E^2(\Omega)$ for Smirnov class on Ω (analogue of Hardy space H^2).

Upper bound theorem (Garabedian 1949, Havinson 1964)

$$\gamma(K) = \min\Big\{ \frac{1}{2\pi} \int_{\partial\Omega} |g^*(z)|^2 \, |dz| : g \in E^2(\Omega),\ g(\infty) = 1 \Big\}$$

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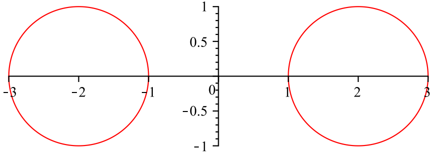
- Theorems are well suited to numerical calculations.
- Extremal g, h expressible via Szegő kernel and Ahlfors function

Example 1: K = union of two disks



$$\gamma(K) = \sqrt{3}(\theta_2(7 - 4\sqrt{3}))^2 \approx 1.8755950190971197289$$

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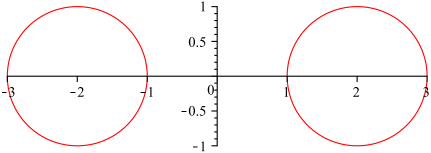

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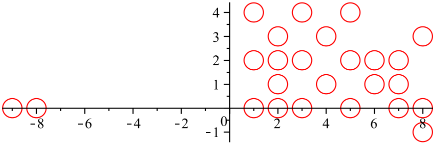
Numerical results

poles/disk	lower bound	upper bound
1	1.875000000000000	1.882812500000000
5	1.875593064023693	1.875619764386366
9	1.875595017927203	1.875595038756883
13	1.875595019096871	1.875595019097141
17	1.875595019097112	1.875595019097164

Example 2: K = union of 25 disjoint disks of equal radius



Exact value of $\gamma(K)$ is unknown

Numerical results

poles/disk	lower bound	upper bound	seconds
1	4.073652478223290	4.219704181009330	0.2
5	4.148169157685863	4.148514554979665	3.7
9	4.148331342401185	4.148332498165111	11.6
13	4.148331931858607	4.148331938572625	24.8
17	4.148331934292544	4.148331934334756	41.3

Example 3: K = square

K = square with vertices at $\pm 1, \pm i$

$$\gamma(K) = \sqrt{2}\Gamma(1/4)^2/(4\pi^{3/2}) \approx 0.8346268416740731863 0.$$

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Numerical results with $g, h \in \text{span}\{z^{-j} : j = 1, \ldots, n\}$

n	lower bound	upper bound
10	0.761941423753061	0.881014562149127
15	0.770723484232218	0.877175902241141
20	0.776589045256849	0.872341829081944
25	0.784189460107018	0.870656623669828
30	0.786857803378602	0.869257904380382
35	0.789068961951613	0.868068649269412
40	0.790942498354322	0.866133165258689

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Numerical results

$$g, h \in \text{span}\{z^{-j},\ z^{-j}(1 - a/z)^{-1/6} : a = \pm 1, \pm i,\ j = 1, \ldots, n\}$$

n	lower bound	upper bound
2	0.834566926465074	0.835066810881929
3	0.834609482283050	0.834678782816948
4	0.834622127643984	0.834628966618492
5	0.834626255962448	0.834627566559480
6	0.834626584020641	0.834627152182154

Example 4: K = rational lemniscate

Let $a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{C}$ and consider

$$K := \left\{ z \in \mathbb{C} : \left| \sum_{j=1}^{n} \frac{a_j}{z - b_j} \right| \geq 1 \right\}$$

Sets like this arise in the investigation of rational Ahlfors functions.

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Theorem (Fortier Bourque–Younsi, 2013)

Assume that K has n components. If a_j are positive and b_j are collinear, then $\gamma(K) = \sum_{j=1}^{n} a_j$.

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What if the b_j are not collinear?

Example 4: K = rational lemniscate (continued)

$$K := \left\{ z \in \mathbb{C} : \left| \sum_{j=1}^{n} \frac{a_j}{z - b_j} \right| \geq 1 \right\}$$

Try $(a_1, a_2, a_3) = (0.4,\ 0.4,\ 0.4)$ and $(b_1, b_2, b_3) = (0,\ 6,\ 1 + i)$.
In this case K has 3 components.

poles/component	lower bound	upper bound
1	1.12585372303575	1.20326750210102
2	1.19741663290495	1.20135320069717
3	1.19938056790033	1.20052466544882
4	1.20021905943941	1.20042627766666
5	1.20032146071966	1.20038739948130
6	1.20036147269825	1.20037878341617
7	1.20037045632015	1.20037593451228

So $\gamma(K) \neq \sum_{j=1}^{n} a_j$ in this case.

The subadditivity problem

Theorem (Tolsa, 2003)

$\gamma(K \cup L) \leq A(\gamma(K) + \gamma(L))$ where A is an absolute constant.

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- Yes if $\gamma(K) = 0$
- Yes if $K, L \subset \mathbb{R}$ (Pommerenke, 1960)
- Yes if K, L disjoint connected sets (Suita, 1994)

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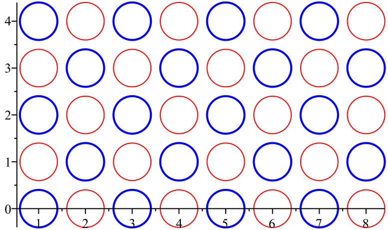
Proposition (Younsi–R, based on results of Melnikov)

Subadditivity holds provided it holds for $K = \cup_1^n K_j$ *and* $L = \cup_1^m L_j$, *where* $K_1, \dots, K_n, L_1, \dots, L_m$ *are disjoint disks of equal radius.*

Graph of $\gamma(K \cup L)/(\gamma(K) + \gamma(L))$ against r



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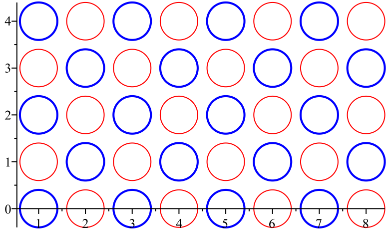

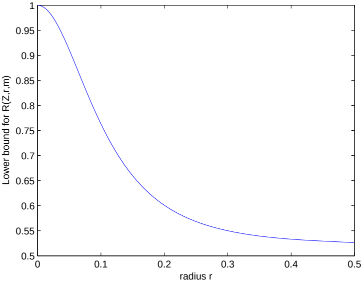

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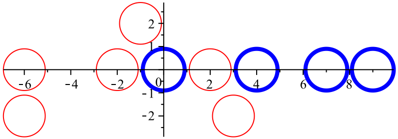

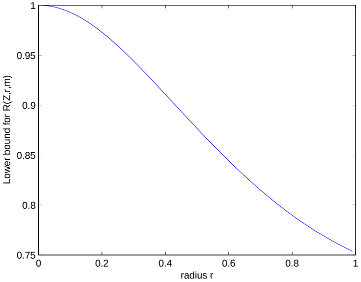

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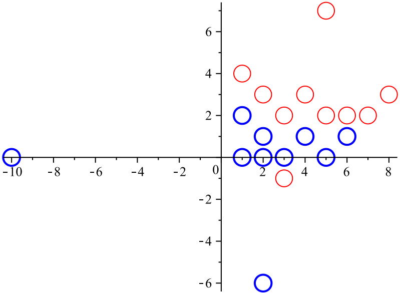

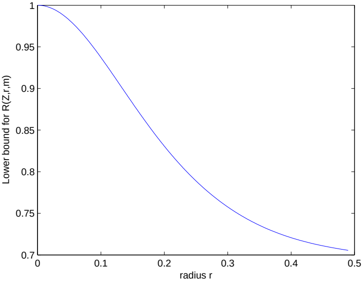

A conjecture

Notation: Let $z_1, \dots, z_n$ and $w_1, \dots, w_m$ be distinct points in $\mathbb{C}$ and let δ be the minimum separation between them. For $r > 0$, set

$$K_r := \cup_{j=1}^n \overline{D}(z_j, r) \qquad \text{and} \qquad L_r := \cup_{j=1}^m \overline{D}(w_j, r).$$

Conjecture (Younsi–R)

$R(r) := \dfrac{\gamma(K_r \cup L_r)}{\gamma(K_r) + \gamma(L_r)}$ *is a decreasing function for* $0 < r < \delta/2$.

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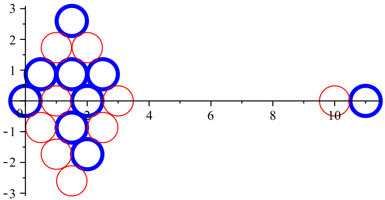
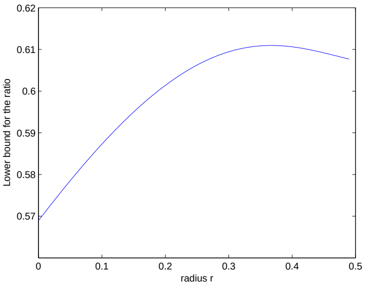
- $R(r) = 1 - ar^2 + O(r^3)$ as $r \to 0$, for some $a > 0$ (Melnikov)
- If conjecture is true, then γ is subadditive.
- Conjecture is true for $n = m = 1$ (proof via elliptic functions).

A cautionary example: where only two radii vary





Tack så mycket!