QUASINILPOTENT EQUIVALENCE IN BANACH ALGEBRAS

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Let A be a Banach algebra and $a, b \in A$. Consider operators L_a and R_b defined on A as follows:

$$L_a x = ax \quad \text{and} \quad R_b x = xb$$

for all $x \in A$.

Definition 1 *Let A be a Banach algebra and $a, b \in A$. Define*

$$\rho(a,b) = \limsup_n ||(L_a - R_b)^n 1||^{1/n}.$$

- In general the numbers $\rho(a,b)$ and $\rho(b,a)$ are different.

- If a and b commute then

$$\rho(a,b) = \rho(b,a) = r(a-b).$$

Definition 2 • *Let A be a Banach algebra and $a, b \in A$. Define*

$$d(a, b) = \max\{\rho(a, b), \rho(b, a)\}.$$

The function d is called the **spectral semidistance** *from a to b.*

- *The elements a and b are called* **quasinilpotent equivalent** *if $d(a, b) = 0$.*

$$\rho(a,b) = \limsup_n ||(L_a - R_b)^n 1||^{1/n} = 0 \quad \Rightarrow \quad ?$$

and

$$d(a,b) = \max\{\rho(a,b), \rho(b,a)\} = 0 \quad \Rightarrow \quad ?$$

Proposition 1 *Let A be a Banach algebra and a, $b \in A$. If $\rho(a,b) = 0$ then $r(a) = r(b)$.*

Theorem 1 *Let A be a Banach algebra and a, $b \in A$. If $d(a,b) = 0$ then $\sigma(a) = \sigma(b)$.*

$$? \quad \Rightarrow \quad d(a,b) = 0.$$

- All quasinilpotent elements in A are quasinilpotent equivalent.

- If $a, b \in A$ and $\sigma(a) = \sigma(b) = \{\lambda\}$ for some $\lambda \in \mathbb{C}$ then $d(a, b) = 0$.

Proposition 2 *Let* A *be a Banach algebra and* $a, b \in A$*. If* $a - b$ *is a commuting quasinilpotent then* $d(a, b) = 0$*.*

Theorem 2 *Let* A *be a Banach algebra and* $a, b \in A$*. Then* $ab = ba$ *and* $d(a, b) = 0$ *if and only if* $a - b$ *is a commuting quasinilpotent.*

Theorem 3 *Let A be a Banach algebra and $a, b \in A$. Then $ab = ba$ and $d(a, b) = 0$ if and only if $a - b$ is a commuting quasinilpotent.*

Example 1 *Let X be a Banach space and $Y = X \oplus X$. Define operators T and S on Y by $T(x_1, x_2) = (0, -x_1)$ and $S(x_1, x_2) = (x_2, 0)$ for all $(x_1, x_2) \in Y$. Then $d(S, T) = 0$, $ST \neq TS$ and $S - T$ is not a commuting quasinilpotent.*

Let A be a semisimple Banach algebra and $a \in A$. Define the **rank** of a by

$$\text{rank}(a) = \sup_{x \in A} \#(\sigma(ax)\backslash\{0\}).$$

An element A is said to be of **maximal finite rank** if

$$\text{rank}(a) = \#(\sigma(a)\backslash\{0\}).$$

Theorem 4 *Let A be a Banach algebra and $a, b \in A$. Then $ab = ba$ and $d(a, b) = 0$ if and only if $a - b$ is a commuting quasinilpotent.*

Theorem 5 *Let A be a semisimple Banach algebra and $0 \neq a \in A$ a maximal finite rank element and $b \in A$. Then $d(a, b) = 0$ if and only if $a - b$ is a commuting quasinilpotent.*

Theorem 6 *Let A be a semisimple Banach algebra with $a, b \in A$. If both a and b are of maximal finite rank and $d(a, b) = 0$ then $a = b$.*

Theorem 7 *Let A be a semisimple Banach algebra and $0 \neq a \in A$ a maximal finite rank element and $b \in A$. Then $d(a,b) = 0$ if and only if $a - b$ is a commuting quasinilpotent.*

Theorem 8 (Brits) *Let A be a semisimple Banach algebra with $a \in \mathrm{Soc}\, A$ and suppose a assumes its rank on* $\mathrm{comm}\,(a)$. *If $b \in A$ then $d(a,b) = 0$ if and only if $a - b$ is a commuting quasinilpotent.*

- a assumes its rank on comm(a) if there is $y \in$comm(a) such that rank(a) $= \#(\sigma(ay)\backslash\{0\})$.

- a is of maximal finite rank if rank(a) $= \#(\sigma(a)\backslash\{0\}) = \#(\sigma(a \cdot 1)\backslash\{0\})$.

Theorem 9 *Let A be a semisimple Banach algebra with $a, b \in A$. If both a and b are of maximal finite rank and $d(a, b) = 0$ then $a = b$.*

Theorem 10 (Brits) *Let A be a semisimple Banach algebra with $a, b \in$ Soc A. If a and b assume their respective ranks on comm(a) and comm(b) and if $d(a, b) = 0$ then $a = b$.*

Theorem 11 *Let A be a C^*- algebra and let a be a normal element of A with finite spectrum. If $b \in A$, then $d(a,b) = 0$ if and only if $a - b$ is a commuting quasinilpotent.*

Theorem 12 (Brits) *Let A be a C^*-algebra and suppose both a and b are normal and suppose 0 is the only possible accumulation point of $\sigma(a)$. If $d(a,b) = 0$ then $a = b$.*