Contractive idempotents on locally compact quantum groups

(joint work with Matthias Neufang, Adam Skalski and Nico Spronk)

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Contractive idempotent measures

$M(G)$ is the measure algebra of a locally compact group G with convolution product:

$$\langle \mu \star \nu, f \rangle = \iint f(st)\, d\mu(s)\, d\nu(t) \qquad \mu, \nu \in M(G); f \in C_0(G).$$

Contractive idempotents: $\mu \in M(G)$ such that $\mu \star \mu = \mu$ and $\|\mu\| = 1$.

Theorem (Greenleaf '65)

Contractive idempotents in $M(G)$ are the measures of the form

$$\chi \, dm_H$$

where m_H is the Haar measure of a compact subgroup H of G and χ is a continuous character of H.

Theorem (Cohen '60)

All idempotent measures on a locally compact abelian group are generated from elementary idempotents $\chi \, dm_H$.

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Contractive idempotents in the Fourier–Stieltjes algebra

The Fourier–Stieltjes algebra $B(G)$ consists of coefficient functions of unitary representations of G.

- $B(G) \cong C^*(G)^*$
- if G abelian, $B(G) \cong M(\widehat{G})$ via Fourier–Stieltjes transform

Theorem (Ilie–Spronk '05)

Contractive idempotents in $B(G)$ are precisely the functions of the form

$$1_C$$

where C is a coset of an open subgroup.

Locally compact quantum groups (Kustermans–Vaes)

A locally compact quantum group $\mathbb{G}$ is determined by

- a C*-algebra $C_0(\mathbb{G})$
- a comultiplication $\Delta\colon C_0(\mathbb{G}) \to M(C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta \qquad \text{(coassociativity)}$$

- left and right Haar weights ϕ and ψ.

Commutative case $\mathbb{G} = G$

- $C_0(\mathbb{G}) = C_0(G)$,
- For $f \in C_0(G)$ and $s, t \in G$,

$$\Delta(f)(s,t) = f(st).$$

- ϕ and ψ are integrations w.r.t. the left and right Haar measures

Co-commutative case

Co-commutative case $\mathbb{G} = \widehat{G}$

- $C_0(\mathbb{G}) = C_r^*(G)$ the reduced group C*-algebra
- $\Delta(\lambda(s)) = \lambda(s) \otimes \lambda(s), \quad \lambda$ is the left regular representation; $s \in G$
- $\phi = \psi$ is the Plancherel weight (for discrete G, $\phi(a) = \langle a\delta_e, \delta_e \rangle$)
- we shall concentrate on coamenable LCQGs which in this case means that G is amenable and hence $C_r^*(G) = C^*(G)$

Common form

Contractive idempotents in $B(G)$ (G amenable)

$$1_C = 1_{sH} = 1_H(s^{-1}\cdot) = 1_H\lambda_{s^{-1}}$$

- 1_H is an idempotent state (i.e. positive idempotent)
- $\lambda_{s^{-1}}$ is a group-like unitary in $M(C^*(G))$: $\Delta(\lambda_{s^{-1}}) = \lambda_{s^{-1}} \otimes \lambda_{s^{-1}}$

Contractive idempotents in $M(G)$

$$\chi dm_H$$

- dm_H is an idempotent state (here an idempotent probability measure)
- χ is a group-like unitary in $C(H)$, $\Delta_H(\chi) = \chi \otimes \chi$.

We may view χ as an element in $C_0(G)$ that is *group-like unitary on the support of* dm_H.

Characterisation of contractive idempotents

Let $\mathbb{G}$ be a locally compact quantum group.
$M(\mathbb{G}) := C_0(\mathbb{G})^*$ is a Banach algebra under

$$\mu \star \nu = (\mu \otimes \nu)\Delta$$

We suppose throughout that $\mathbb{G}$ is coamenable: there is a unit in $M(\mathbb{G})$.
A contractive idempotent on $\mathbb{G}$ is $\omega \in M(\mathbb{G})$ such that $\omega \star \omega = \omega$ and $\|\omega\| = 1$.

Theorem (NSSS)

Contractive idempotents in $M(\mathbb{G})$ are precisely the functionals in $M(\mathbb{G})$ of the form $\omega = |\omega|(v\cdot)$ where $|\omega|$ is an idempotent state and $v \in C_0(\mathbb{G})$ such that

$$\Delta(v) - v \otimes v \in N_{|\omega| \otimes |\omega|}.$$

Here $N_\sigma = \{ a \in A;\ \sigma(a^* a) = 0 \}$ for σ a positive functional on a C*-algebra A.

Special case: Haar idempotents

Recall that $|\omega|$ is a Haar idempotent if $|\omega| = \phi_{\mathbb{H}} \circ \pi$ where

- $\pi \colon C_0(\mathbb{G}) \to C(\mathbb{H})$ is a surjective $*$-homomorphism such that $\Delta_{\mathbb{H}}\pi = (\pi \otimes \pi)\Delta_{\mathbb{G}}$
- $\phi_{\mathbb{H}}$ is the Haar state of a compact quantum subgroup $\mathbb{H}$ of $\mathbb{G}$

Corollary (NSSS)

Let ω be a contractive idempotent on $\mathbb{G}$ such that $|\omega| = \phi_{\mathbb{H}} \circ \pi$ is a Haar idempotent. Then there is a group-like unitary $u \in C(\mathbb{H})$ such that

$$\omega = \phi_{\mathbb{H}}(u\pi(\cdot)).$$

This is in perfect agreement with Greenleaf's result.

TROs and conditional expectations

A TRO is a closed subspace T of $B(H, K)$ such that

$$ab^*c \in T \quad \text{whenever } a, b, c \in T.$$

Linking algebra (C*-algebra):

$$A_T = \begin{bmatrix} \langle TT^* \rangle & T \\ T^* & \langle T^*T \rangle \end{bmatrix} \subseteq B(K \oplus H).$$

A TRO conditional expectation is a contractive projection P from a TRO T onto a sub-TRO $X \subseteq T$. Then

$$P(ax^*y) = P(a)x^*y, \qquad P(xa^*y) = xP(a)^*y, \qquad P(xy^*a) = xy^*P(a),$$

whenever $a \in T$ and $x, y \in X$ (Effros–Ozawa–Ruan '01).

TRO structures associated with ω

Every $\omega \in M(\mathbb{G}) = C_0(\mathbb{G})^*$ induces convolution operators

$$L_\omega \colon C_0(\mathbb{G}) \to C_0(\mathbb{G}), \quad L_\omega(a) = (\omega \otimes \mathrm{id})\Delta(a),$$
$$R_\omega \colon C_0(\mathbb{G}) \to C_0(\mathbb{G}), \quad R_\omega(a) = (\mathrm{id} \otimes \omega)\Delta(a).$$

Theorem (NSSS)

If ω is a contractive idempotent,

- *$L_\omega(C_0(\mathbb{G}))$ is a sub-TRO of $C_0(\mathbb{G})$*
- *L_ω is a TRO conditional expectation.*

Theorem (Franz–Skalski '09; S–Skalski '12)

If ω is an idempotent state,

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If ω is an idempotent state,

- $L_\omega(C_0(\mathbb{G}))$ *is a C*-subalgebra of* $C_0(\mathbb{G})$
- L_ω *is a conditional expectation.*

Expectations and weights

Let η be a weight on a C*-algebra A. A C*-subalgebra $X \subseteq A$ is η-expected if there is a conditional expectation $P\colon A \to X$ preserving η in the sense that $\eta(P(a)) = \eta(a)$.

Let ψ be the right Haar weight of $\mathbb{G}$. Define $\psi^{(2)}\colon M_2(C_0(\mathbb{G}))_+ \to [0, \infty]$ by

$$\psi^{(2)} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \psi(a_1) + \psi(a_4).$$

Then $\psi^{(2)}$ is a densely defined, faithful, lower semicontinuous weight on $M_2(C_0(\mathbb{G}))$.

Correspondence result for unimodular LCQG

Theorem (NSSS)

Suppose that $\mathbb{G}$ is unimodular ($\phi = \psi$). There is a one-to-one correspondence between

- *contractive idempotents ω on $\mathbb{G}$*
- *nondegenerate sub-TROs $X \subseteq C_0(\mathbb{G})$ such that A_X is $\psi^{(2)}$-expected and right invariant.*

The expectation is given by

$$\begin{bmatrix} L_{|\omega|_r} & L_\omega \\ L_{\overline{\omega}} & L_{|\omega|_l} \end{bmatrix}.$$

$A \subseteq M_2(C_0(\mathbb{G}))$ is right invariant if

$$\begin{bmatrix} R_\mu & R_\mu \\ R_\mu & R_\mu \end{bmatrix}(\mathbf{a}) \in A \qquad \forall \mathbf{a} \in A, \mu \in C_0(\mathbb{G})^*.$$

$X \subseteq T$ is nondegenerate if $\langle XT^*T \rangle = T$ and $\langle TT^*X \rangle = T$.

Correspondence result for those ω with $|\omega|$ Haar

Theorem (NSSS)

There is a one-to-one correspondence between

- *contractive idempotents* ω *on* $\mathbb{G}$ *such that* $|\omega|$ *is a Haar idempotent*
- *nondegenerate sub-TROs* $X \subseteq C_0(\mathbb{G})$ *such that* A_X *is* $\psi^{(2)}$*-expected and right invariant and* $\langle XX^* \rangle$ *is symmetric.*

A right invariant C*-subalgebra A of $C_0(\mathbb{G})$ is symmetric if

$$V^*(1 \otimes a)V \in M\big(\mathcal{K}(\mathrm{L}^2(\mathbb{G})) \otimes A\big) \quad \text{for every } a \in A.$$

Here V is the right multiplicative unitary of $\mathbb{G}$. The symmetry condition goes back to Tomatsu '07 and S '11.