An analogue of Serre's Conjecture and Control Theory

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Theorem $\mathcal{M}_+$ is a projective free ring.

$\mathcal{M}_+ = \{$ complex Borel measures on $[0, +\infty) \}$

Projective free ring ?

Control Theory ?

Projective free?

R is projective free

$\Leftrightarrow$ every finitely generated projective R-module is free

$M \oplus N \simeq R^n$ $\qquad$ $M \simeq R^n$

$\Leftrightarrow$ $\forall\, n \in \mathbb{N}$ $\forall\, P \in R^{n \times n}$ s.t. $P^2 = P$,

$\exists\, S \in R^{n \times n}$ s.t. $S^{-1} P S = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$ for some $k \geq 0$

"Every projection can be diagonalized."

Examples.

(1) $\mathbb{C}$ $\qquad$ $P \in \mathbb{C}^{n \times n}$ s.t. $P^2 = P$

ker P $\qquad$ ran P

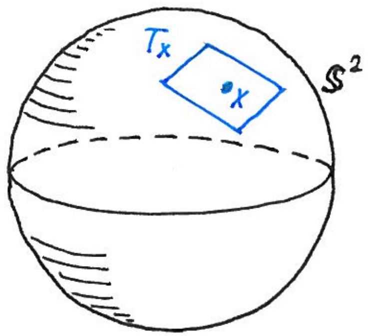
$\underbrace{v_1, \dots, v_k}_{\text{eig. } 0}$ $\qquad$ $\underbrace{v_{k+1}, \dots, v_n}_{\text{eig. } 1}$ $\qquad$ $\left(\begin{array}{l} y = Px \\ Py = P^2 x = Px = y \end{array} \right)$

(2) $k[x_1, \dots, x_n]$ $\qquad$ Serre's Conjecture 1955

(k field) $\qquad$ Quillen/Suslin Theorem 1976

Non-example

$C(\mathbb{S}^2; \mathbb{R})$



Hairy Ball Theorem

$P :=$ projection on T_x, $x \in \mathbb{S}^2$

$$S^{-1} P S = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix}$$

$P v = v$ $\qquad$ ↯

↑

first column of S,

never zero on $\mathbb{S}^2$

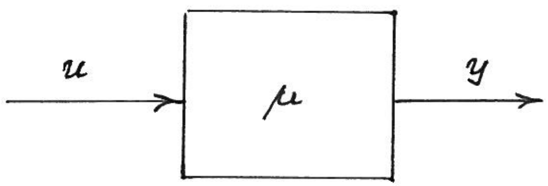
Control Theory

$$u \longmapsto y$$

$$u, y : [0, \infty) \longrightarrow \mathbb{C}$$

$$y = \mu * u$$

output   impulse response   input



$\mathcal{M}_+ := \{ \mu : \mu$ is a complex Borel measure on $\mathbb{R}$ with $\operatorname{supp} \mu \subset [0,\infty) \}$

$(\mathcal{M}_+, +, *, \|\cdot\|_{\mathcal{M}_+})$ is a Banach algebra with $\|\cdot\|_{\mathcal{M}_+}$ given by

$\|\mu\|_{\mathcal{M}_+} = \sup \sum_{n=1}^{\infty} |\mu(E_n)|$ where E_n are all Borel sets s.t.

$\bigcup_{n\geq 1} E_n = [0,\infty)$ and

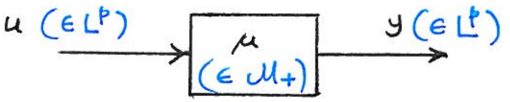
$E_n \neq E_m$ for $n \neq m$.

$\mathcal{M}_+$ is a class of stable impulse responses:

$\mu \in \mathcal{M}_+ \Rightarrow$ "nice" inputs are mapped to "nice" outputs

If $1 \leq p \leq \infty$ and $u \in L^p[0,\infty)$, then $y \in L^p[0,\infty)$.

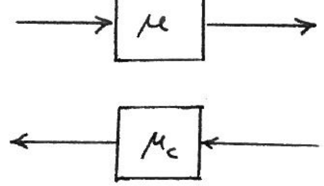
Moreover, $\displaystyle \sup_{0 \neq u \in L^p} \frac{\|y\|_p}{\|u\|_p} \leq \|\mu\|_{\mathcal{M}_+}$.



Stable systems : $\mu \in \mathcal{M}_+$

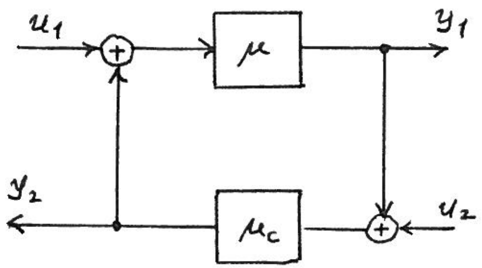
Nonstable systems : $\mu \in \mathbb{F}(\mathcal{M}_+) = \left\{ \frac{n}{d} : n, d \in \mathcal{M}_+, \ d \neq 0 \right\}$.

Stabilization problem : Given $\mu \in \mathbb{F}(\mathcal{M}_+)$,



find $\mu_c \in \mathbb{F}(\mathcal{M}_+)$

s.t. $\begin{bmatrix} \frac{\mu}{1-\mu\mu_c} & \frac{\mu\mu_c}{1-\mu\mu_c} \\ \frac{\mu\mu_c}{1-\mu\mu_c} & \frac{\mu_c}{1-\mu\mu_c} \end{bmatrix} \in \mathcal{M}_+^{2\times 2}$.



Role of projective freeness.

μ has a coprime factorization $\Rightarrow$ μ is stabilizable

$\left(\mu = \frac{n}{d} \right.$ where $n, d \in \mathcal{M}_+$, $d \neq 0$ and $\Leftarrow$ $\left(\mu_c := -\frac{x}{y} \right)$

$\left. \exists \, x, y \in \mathcal{M}_+ \text{ s.t. } nx + dy = \delta \right)$

holds
if $\mathcal{M}_+$ is projective free (A. Quadrat 2003)

Projective freeness of Banach algebras

Theorem (Brudnyi-S., 2010)

$M(R)$ is contractible $\Rightarrow$ R is projective free

$$M(R) = \{\text{multiplicative nontrivial linear functionals on } R\}$$

$$\cap$$

$$\mathcal{L}(R;\mathbb{C}) \quad \text{weak-* topology}$$

Theorem (A. Sasane, 2010)

$M(\mathcal{M}_+)$ is contractible.

Corollaries

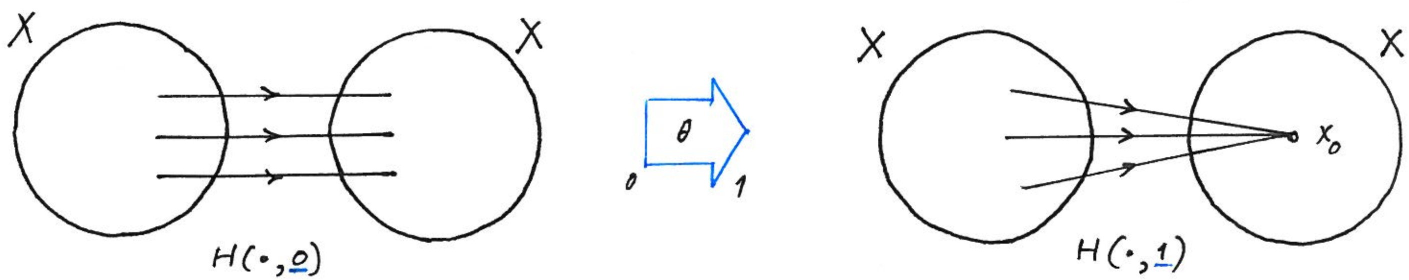
(1) $\mathcal{M}_+$ is projective free.

(2) μ is stabilizable $\Leftrightarrow$ μ has a coprime factorization.

A topological space X is contractible if

$\exists$ $x_0 \in X$ and a continuous $H : X \times \underbrace{[0,1]}_{\theta} \longrightarrow X$ such that

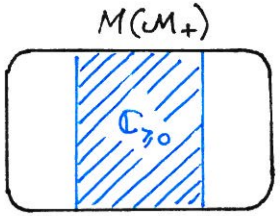
$$\left.\begin{aligned} H(x,0) &= x \\ H(x,1) &= x_0 \end{aligned}\right\} \quad \forall\, x \in X$$



$$\varphi \in M(\mathcal{M}_+)$$

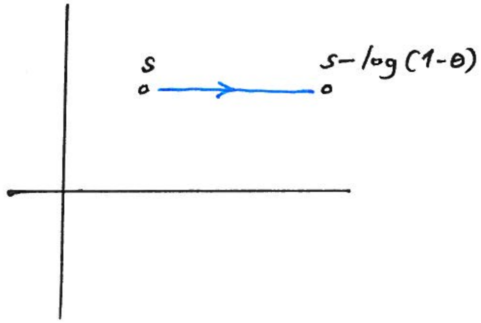
$$\theta \in [0,1]$$

$$\big(H(\varphi,\theta)\big)(\mu) := \varphi(\mu_\theta) \qquad d\mu_\theta(t) = (1-\theta)^t \, d\mu(t)$$



$$H(\underline{s},\theta) \qquad \mu \xrightarrow{\underline{s}} \hat{\mu}(s) = \int_0^\infty e^{-st} \, d\mu(t)$$

$$\begin{aligned}
\big(H(\underline{s},\theta)\big)(\mu) &= \underline{s}(\mu_\theta) \\
&= \int_0^\infty e^{-st} (1-\theta)^t \, d\mu(t) \\
&= \int_0^\infty e^{-(s-\log(1-\theta))t} \, d\mu(t) \\
&= \big(\underline{s-\log(1-\theta)}\big)(\mu).
\end{aligned}$$



$$\theta = 0 \qquad H(\underline{s},0) = \underline{s}$$

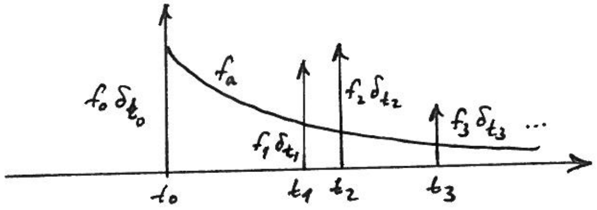
$$\theta \nearrow 1 \qquad -\log(1-\theta) \nearrow +\infty \qquad H(\underline{s},\theta) \longrightarrow \underline{+\infty}$$

$$\mu \xrightarrow{\underline{+\infty}} \lim_{s \to +\infty} \hat{\mu}(s). \qquad \square$$

$R \qquad\qquad\qquad\qquad \subset \mathcal{M}_+$

Banach subalgebra of $\mathcal{M}_+$

closed under $\mu \mapsto \mu_\theta$, $0 \le \theta \le 1$.



Examples :

$$\mathcal{A}_+ := \{\mu \in \mathcal{M}_+ : \mu \text{ does not have a singular nonatomic part}\}$$

$$= \Big\{ f_a + \sum_{k \geq 0} f_k \delta_{t_k} : f_a \in L^1[0,\infty),\ (f_k)_{k\geq 0} \in \ell^1,\ 0 = t_0 < t_1, t_2, t_3, \ldots \Big\}.$$

$$APW_+ := \Big\{ \sum_{k\geq 0} f_k \delta_{t_k} : (f_k)_{k\geq 0} \in \ell^1,\ 0 = t_0 < t_1, t_2, t_3, \ldots \Big\}$$

$$L^1[0,\infty) + \mathbb{C}\delta_0$$