Hardy Algebras, Berezin Transform and Taylor’s Taylor Series

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We study tensor operator algebras (to be defined shortly) and their ultraweak closures: the **Hardy algebras**. We want to study these algebras as **algebras of (operator valued) functions** defined on the representation space of the algebra. More precisely, we are led to consider a **family of functions** defined on a **family of sets**. I shall discuss the “**matricial structure**” of this family of functions and their “**power series**” expansions.

♣ We were inspired by works of J. Taylor, D. Voiculescu, Kaliuzhnyi-Verbovetskyi and Vinnikov and Helton-Klepp-McCullough.
We begin with the following setup:

- $M$ - a $W^*$-algebra.
- $E$ - a $W^*$-correspondence over $M$. This means that $E$ is a bimodule over $M$ which is endowed with an $M$-valued inner product (making it a right-Hilbert $C^*$-module that is self dual). The left action of $M$ on $E$ is given by a unital, normal, *-homomorphism $\varphi$ of $M$ into the ($W^*$-) algebra of all bounded adjointable operators $\mathcal{L}(E)$ on $E$. 

Introduction  The algebras  Representations  Family of functions  without a generator  Maps

Examples

• (Basic Example) $M = \mathbb{C}$, $E = \mathbb{C}^d$, $d \geq 1$.

• $G = (G^0, G^1, r, s)$- a finite directed graph. $M = \ell^\infty(G^0)$, $E = \ell^\infty(G^1)$, $a\xi b(e) = a(r(e))\xi(e)b(s(e))$, $a, b \in M, \xi \in E$

$$\langle \xi, \eta \rangle(v) = \sum_{s(e) = v} \xi(e)\eta(e), \xi, \eta \in E.$$  

• $M$- arbitrary , $\alpha : M \to M$ a normal unital, endomorphism. $E = M$ with right action by multiplication, left action by $\varphi = \alpha$ and inner product $\langle \xi, \eta \rangle := \xi^*\eta$. Denote it $\alpha M$.

• $\Phi$ is a normal, contractive, CP map on $M$. $E = M \otimes \Phi M$ is the completion of $M \otimes M$ with $\langle a \otimes b, c \otimes d \rangle = b^*\Phi(a^*c)d$ and $c(a \otimes b)d = ca \otimes bd$.

Note: If $\sigma$ is a representation of $M$ on $H$, $E \otimes_\sigma H$ is a Hilbert space with $\langle \xi_1 \otimes h_1, \xi_2 \otimes h_2 \rangle = \langle h_1, \sigma(\langle \xi_1, \xi_2 \rangle_E)h_2 \rangle_H$. 
Similarly, given two correspondences $E$ and $F$ over $M$, we can form the (internal) tensor product $E \otimes F$ by setting

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle = \langle f_1, \varphi(\langle e_1, e_2 \rangle_E)f_2 \rangle_F$$

$$\varphi_{E \otimes F}(a)(e \otimes f)b = \varphi_E(a)e \otimes fb$$

and applying an appropriate completion.

In particular we get “tensor powers” $E \otimes^k$.

Also, given a sequence $\{E_k\}$ of correspondences over $M$, the direct sum $E_1 \oplus E_2 \oplus E_3 \oplus \cdots$ is also a correspondence (after an appropriate completion).
For a correspondence $E$ over $M$ we define the Fock correspondence
\[ \mathcal{F}(E) := M \oplus E \oplus E^\otimes 2 \oplus E^\otimes 3 \oplus \cdots \]
For every $a \in M$ define the operator $\varphi_\infty(a)$ on $\mathcal{F}(E)$ by
\[ \varphi_\infty(a)(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_n \]
and $\varphi_\infty(a)b = ab$.
For $\xi \in E$, define the “shift” (or “creation”) operator $T_\xi$ by
\[ T_\xi(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n. \]
and $T_\xi b = \xi b$. So that $T_\xi$ maps $E^\otimes k$ into $E^\otimes (k+1)$. 
**Definition**

(1) The norm-closed algebra generated by $\varphi_\infty(M)$ and \{ $T_\xi : \xi \in E$ \} will be called the **tensor algebra** of $E$ and denoted $T_+(E)$.

(2) The ultra-weak closure of $T_+(E)$ will be called the **Hardy algebra** of $E$ and denoted $H_\infty(E)$.

**Examples**

1. If $M = E = \mathbb{C}$, $F(E) = \ell^2$, $T_+(E) = A(\mathbb{D})$ and $H_\infty(E) = H_\infty(\mathbb{D})$.

2. If $M = \mathbb{C}$ and $E = \mathbb{C}^d$ then $F(E) = \ell^2(\mathbb{F}_d^+)$, $T_+(E)$ is Popescu’s $A_d$ and $H_\infty(E)$ is $F_\infty_d$ (Popescu) or $\mathcal{L}_d$ (Davidson-Pitts). These algebras are generated by $d$ shifts $\{S_i\}$, each $S_i$ is an isometry and $\sum S_i S_i^* \leq I$. 
Every completely contractive representation of $\mathcal{T}_+(E)$ on $H$ is given by a pair $(\sigma, \tilde{\rho})$ where

1. $\sigma$ is a normal representation of $M$ on $H = H_\sigma$. ($\sigma \in N\text{Rep}(M)$)
2. $\tilde{\rho} : E \otimes_\sigma H \to H$ is a contraction that satisfies

$$\tilde{\rho}(\varphi(\cdot) \otimes I_H) = \sigma(\cdot)\tilde{\rho}.$$ 

We write $\sigma \times \tilde{\rho}$ for the representation and we have

$$(\sigma \times \tilde{\rho})(\varphi_\infty(a)) = \sigma(a) \text{ and } (\sigma \times \tilde{\rho})(T_\xi) h = \tilde{\rho}(\xi \otimes h) \text{ for } a \in M, \xi \in E \text{ and } h \in H.$$ 

Write $\mathcal{I}(\varphi \otimes I, \sigma)$ for the intertwining space and $\mathbb{D}(0,1,\sigma)$ for the open unit ball there. Thus the c.c. representations of the tensor algebra are parametrized by the family $\{\overline{\mathbb{D}(0,1,\sigma)}\}_{\sigma \in N\text{Rep}(M)}$. 
Examples

(1) \( M = E = \mathbb{C} \). So \( \mathcal{T}_+(E) = A(\mathbb{D}) \), \( \sigma \) is the trivial representation on \( H \), \( E \otimes H = H \) and \( \mathbb{D}(0, 1, \sigma) \) is the (open) unit ball in \( B(H_\sigma) \).

(2) \( M = \mathbb{C}, \ E = \mathbb{C}^d \). \( \mathcal{T}_+(E) = A_d \) (Popescu’s algebra) and \( \mathbb{D}(0, 1, \sigma) \) is the (open) unit ball in \( B(\mathbb{C}^d \otimes H, H) \). Thus the c.c. representations are parameterized by row contractions \((T_1, \ldots, T_d)\).

(3) \( M \) general, \( E =_\alpha M \) for an automorphism \( \alpha \). 
\( \mathcal{T}_+(E) = \) the analytic crossed product. 
The intertwining space can be identified with 
\( \{X \in B(H) : \sigma(\alpha(T))X = X\sigma(T), T \in B(H)\} \) and the c.c. representations are \( \sigma \times \_3 \) where \( \_3 \) is a contraction there.
Representations of $\mathcal{H}^\infty(E)$

The representations of $\mathcal{H}^\infty(E)$ are given by the representations of $\mathcal{I}_+(E)$ that extend to an ultraweakly continuous representations of $\mathcal{H}^\infty(E)$.

For a given $\sigma$, we write $\mathcal{AC}(\sigma)$ for the set of all $\mathfrak{z} \in \mathbb{D}(0,1,\sigma)$ such that $\sigma \times \mathfrak{z}$ is a representation of $\mathcal{H}^\infty(E)$.

We have

Theorem

$$\mathbb{D}(0,1,\sigma) \subseteq \mathcal{AC}(\sigma) \subseteq \overline{\mathbb{D}(0,1,\sigma)}.$$ 

Example

When $M = E = \mathbb{C}$, $\mathcal{H}^\infty(E) = \mathcal{H}^\infty(\mathbb{D})$ and $\mathcal{AC}(\sigma)$ is the set of all contractions in $\mathcal{B}(\mathcal{H}_\sigma)$ that have an $\mathcal{H}^\infty$-functional calculus.
**Example**

**Induced representations:** Fix a normal representation $\pi$ of $M$ on $K$, let $H = F(E) \otimes_{\pi} K$ and define the representation of $H^\infty(E)$ on $H$ by $X \mapsto X \otimes I_K$.

It is $\sigma \times \zeta$ for $\sigma(a) = \varphi_\infty(a) \otimes I_K$ and $\zeta(\xi \otimes h) = (T_\xi \otimes I_K)h$.

Note that $\|\zeta\| = 1$ and $\zeta \in AC(\sigma)$.

When $\pi$ is faithful of infinite multiplicity we write $\sigma_0 \times s_0$ for the induced representation. It is essentially independent of $\pi$ and is a universal generator in the following sense.
Universal induced representation

Theorem

Let $\sigma \times \mathcal{F}$ be a c.c. representation of $\mathcal{I}_+(E)$ on $H$. Then the following are equivalent.

1. The representation $\sigma \times \mathcal{F}$ extends to a c.c. ultra weakly continuous representation of $H^\infty(E)$ (that is, $\mathcal{F} \in AC(\sigma)$).

2. $H = \bigvee \{\text{Ran}(C) : C \in \mathcal{I}(\sigma_0 \times s_0, \sigma \times \mathcal{F})\}$.

Here $\mathcal{I}(\sigma_0 \times s_0, \sigma \times \mathcal{F})$ is the space of all maps from $H_{\sigma_0}$ to $H_\sigma$ that intertwine the representations $\sigma_0 \times s_0$ and $\sigma \times \mathcal{F}$.

Partial results: Douglas (69), Davidson-Li-Pitts (05).
The families of functions

Given $F \in H^\infty(E)$, we define a family $\{\hat{F}_\sigma\}_{\sigma \in NRep(M)}$ of (operator valued) functions. Each function $\hat{F}_\sigma$ is defined on $\mathcal{AC}(\sigma)$ (or on $\mathbb{D}(0, 1, \sigma)$) and takes values in $B(H_\sigma)$:

$$\hat{F}_\sigma(\zeta) = (\sigma \times \zeta)(F).$$

Here $NRep(M)$ is the set of all normal representations of $M$. Note that the family of domains (either $\{\mathcal{AC}(\sigma)\}$ or $\{\mathbb{D}(0, 1, \sigma)\}$) is a matricial family in the following sense.

**Definition**

A family of sets $\{U(\sigma)\}_{\sigma \in NRep(M)}$, with $U(\sigma) \subseteq \mathcal{I}(\varphi \otimes I, \sigma)$, satisfying $U(\sigma) \oplus U(\tau) \subseteq U(\sigma \oplus \tau)$ is called a **matricial family of sets**.
Definition

Suppose \( \{ \mathcal{U}(\sigma) \}_{\sigma \in \text{NRep}(M)} \) is a matricial family of sets and suppose that for each \( \sigma \in \text{NRep}(M) \), \( f_\sigma : \mathcal{U}(\sigma) \rightarrow B(H_\sigma) \) is a function. We say that \( f := \{ f_\sigma \}_{\sigma \in \text{NRep}(M)} \) is a matricial family of functions in case for every \( z \in \mathcal{U}(\sigma) \), every \( w \in \mathcal{U}(\tau) \) and every \( C \in \mathcal{I}(\sigma \times z, \tau \times w) \), we have

\[
Cf_\sigma(z) = f_\tau(w)C
\] (1)

Theorem

For every \( F \in H^\infty(E) \), the family \( \{ \hat{F}_\sigma \} \) is is a matricial family (on \( \{ \mathcal{AC}(\sigma) \} \)).

Conversely, if \( f = \{ f_\sigma \}_{\sigma \in \text{NRep}(M)} \) is a matricial family of functions, with \( f_\sigma \) defined on \( \mathcal{AC}(\sigma) \) and mapping to \( B(H_\sigma) \), then there is an \( F \in H^\infty(E) \) such that \( f \) is the Berezin transform of \( F \), i.e., \( f_\sigma = \hat{F}_\sigma \) for every \( \sigma \).
**Notation:** For $\mathcal{z} \in \mathcal{I}(\mathcal{v} \otimes I, \sigma)$ and $k \geq 1$, 
$Z_k(\mathcal{z}) = \mathcal{z}(I_E \otimes \mathcal{z}) \cdots (I_{E \otimes k-1} \otimes \mathcal{z}) \in \mathcal{I}(\mathcal{v}_{E \otimes k} \otimes I, \sigma)$.

For a sequence $\theta = \{\theta_k\}$, with $\theta_k \in E \otimes^k$, 
$L_{\theta_k} : H \rightarrow E \otimes^k \otimes H, \quad L_{\theta_k} h = \theta_k \otimes h$

and $R(\theta) = (\lim \sup_{k \to \infty} \|\theta_k\|_1^k)^{-1}$. (Popescu)

**Theorem**

If $f = \{f_{\sigma}\}_{\sigma \in N\text{Rep}(M)}$ is a family of functions, with $f_{\sigma}$ mapping $\mathbb{D}(0, 1, \sigma)$ to $B(H_\sigma)$, then $f$ is a matricial family of functions if and only if there is a formal tensor series $\theta$ with $R(\theta) \geq 1$ such that $f$ is the family of tensorial power series determined by $\theta$; that is,

$$f_{\sigma}(\mathcal{z}) = \sum_{k \geq 0} Z_k(\mathcal{z}) L_{\theta_k}.$$  

Moreover, $f = \hat{F}$ for some $F \in H^\infty(E)$ if and only if

$$\sup\{\|f_{\sigma}(\mathcal{z})\| \mid \sigma \in N\text{Rep}(M), \mathcal{z} \in \mathbb{D}(0, 1, \sigma)\} < \infty. \quad (2)$$
Function theory without a generator

Now we fix an additive subcategory $\Sigma$ of $NRep(M)$ that do not necessarily contain a special generator. Then

**Theorem**

Suppose that $f = \{f_\sigma\}_{\sigma \in \Sigma}$ is a matricial family of functions defined on $\{\mathbb{D}(0,1,\sigma)\}$ that is locally uniformly bounded in the sense that for each $r < 1$, $\sup_{\sigma \in \Sigma} \sup_{z \in \mathbb{D}(0,r,\sigma)} \|f_\sigma(z)\| < \infty$. Then:

1. Each $f_\sigma$ is Frechet analytic on $\mathbb{D}(0,1,\sigma)$ and

   $$f_\sigma(z) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f_\sigma(0)(z).$$

2. If the subcategory is full and if each $\sigma \in \Sigma$ is faithful, then there is $\theta = \{\theta_k\}$ with $R(\theta) \geq 1$ and

   $$f_\sigma(z) = \sum_{k \geq 0} Z_k(z) L_{\theta_k}$$
Now we discuss another expansion: the Taylor-Taylor series. We first need the following.

**Theorem**

Let \( f = \{f_\sigma\} \in \Sigma \) be a matricial family of functions defined on a matricial family \( \{U(\sigma)\} \in \Sigma \) where \( \Sigma \) is an additive subcategory of \( N\text{Rep}(M) \). Suppose \( \sigma, \tau \in \Sigma, \ z \in U(\sigma), \ w \in U(\sigma) \) and \( u \in I(\varphi \otimes_\tau I, \sigma) \) are such that \( \left( \begin{array}{cc} z & u \\ 0 & w \end{array} \right) \in U(\sigma \oplus \tau) \). Then there is an operator \( \Delta f_{\sigma, \tau}(z, w)(u) \in B(H_\tau, H_\sigma) \) such that

\[
\begin{array}{ccc}
\sigma \oplus \tau & (z & u) \\
0 & 0 & w
\end{array}
\begin{array}{c}
\Delta f_{\sigma, \tau}(z, w)(u)
\end{array}
\begin{array}{c}
f_\sigma(z)
\end{array}
\begin{array}{c}
f_\tau(w)
\end{array}
\]

Also, the map \( u \mapsto \Delta f_{\sigma, \tau}(z, w)(u) \) is linear.
Similarly, we write $\Delta^n f_{\sigma_0, \sigma_1, \ldots, \sigma_n}(z_0, \ldots, z_n)(u_1, \ldots, u_n)$ for the operator on the top-right corner of the matrix obtained by

$$f_{\sigma_0 \oplus \sigma_1 \oplus \cdots \oplus \sigma_n}(\begin{pmatrix} z_0 & u_1 & 0 & \cdots & 0 \\ 0 & z_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & z_{n-1} & u_n \\ 0 & \cdots & \cdots & 0 & z_n \end{pmatrix})$$ (3)

This map is multilinear in $u_1, \ldots, u_n$.

**Definition**

The function $\Delta^n f_{\sigma_0, \sigma_1, \ldots, \sigma_n}(z_0, \ldots, z_n)$ of $u_1, u_2, \cdots, u_n$ defined above will be called the $n^{th}$-order Taylor difference operator determined by $z_0, z_1, \ldots, z_n$. If $z_0 = z_1 = \cdots = z_n = z$, we call $\Delta^n f_{\sigma, \sigma, \ldots, \sigma}(z, z, \ldots, z) := \Delta^n f_\sigma(z)$ the $n^{th}$-order Taylor derivative of $f_\sigma$ at $z$. 
Theorem (T-T Series)

Let \( f = \{ f_\sigma \}_{\sigma \in \Sigma} \) be a matricial family of functions defined on a matricial disc \( \mathbb{D}(0, r) \) (= \( \{ \mathbb{D}(0, r, \sigma) \}_{\sigma} \)) and suppose that \( f \) is locally uniformly bounded. Then:

1. Each \( f_\sigma \) is Frechet differentiable in \( \zeta, \zeta \in \mathbb{D}(0, r, \sigma) \), and
   \[
   f'_\sigma(\zeta)(\omega) = \Delta f(\zeta)(\omega).
   \]

2. \[
   D^k f_\sigma(0)(\omega) = k! \Delta^k f_\sigma(0)(\omega).
   \]

3. Each \( f_\sigma \) may be expanded on \( \mathbb{D}(0, r, \sigma) \) as
   \[
   f_\sigma(\zeta) = \sum_{k=0}^{\infty} \Delta^k f_\sigma(0)(\zeta, \ldots, \zeta),
   \]
   (4)

where the series converges absolutely and uniformly on every disc \( \mathbb{D}(0, r_0, \sigma) \) with \( r_0 < r \).
Suppose that $E$ and $F$ are two $W^*$-correspondences over $M$ and that $f = \{ f_\sigma \}_\sigma$ is a family of maps, with $f_\sigma : AC(\sigma, E) \to AC(\sigma, F)$. Then $f$ is a matricial family of maps (that is, preserves intertwiners) if and only if there is an ultraweakly continuous homomorphism $\alpha : H^\infty(F) \to H^\infty(E)$ such that for every $\hat{z} \in AC(\sigma, E)$ and every $\hat{Y} \in H^\infty(F)$,

$$\hat{\alpha}(\hat{Y})(\hat{z}) = \hat{Y}(f_\sigma(\hat{z})).$$

(5)
Given two correspondences $E, F$ over $M$, we will write $M\mathcal{L}_M(E, F)$ for the maps in $\mathcal{L}(E, F)$ that are bimodule maps. That is, $T \in \mathcal{L}(E, F)$ lies in $M\mathcal{L}_M(E, F)$ if and only if $T(\varphi_E(a)\xi b) = \varphi_F(a)T(\xi)b$, for all $a, b \in M$.

**Theorem**

Let $E$ and $F$ be two $W^*$-correspondences over the same $W^*$-algebra, $M$, and suppose $\Sigma$ is a full additive subcategory of $N\text{Rep}(M)$ whose objects are all faithful representations of $M$. If $f = \{f_\sigma\}_{\sigma \in \Sigma}$ is a matricial family of maps, mapping a disc $\mathbb{D}(0, r, \sigma, E)$ to a disc $\mathbb{D}(0, R, \sigma, F)$, then there is a uniquely defined sequence of maps $\{\mathcal{D}^k f\}_{k=0}^\infty$, where for each $k$, $\mathcal{D}^k f$ lies in $M\mathcal{L}_M(F, E \otimes^k)$, such that for every $\zeta \in \mathbb{D}(0, r, \sigma, E)$,

$$f_\sigma(\zeta) = f_\sigma(0) + \sum_{k \geq 1} Z_k(\zeta)(\mathcal{D}^k f \otimes I_{H_\sigma}). \quad (6)$$


Thank You!