Hardy Algebras, Berezin Transform and Taylor's Taylor Series

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Introduction

We study tensor operator algebras (to be defined shortly) and their ultraweak closures: the **Hardy algebras**.
We want to study these algebras as **algebras of (operator valued) functions** defined on the representation space of the algebra.
More precisely, we are led to consider a **family of functions** defined on a **family of sets**.
I shall discuss the **"matricial structure"** of this family of functions and their **"power series" expansions**.

♣ We were inspired by works of J. Taylor, D. Voiculescu, Kaliuzhnyi-Verbovetskyi and Vinnikov and Helton-Klepp-McCullough.

The Setup

We begin with the following setup:

- M - a W^*-algebra.
- E - a W^*-correspondence over M. This means that E is a **bimodule** over M which is endowed with an M-**valued inner product** (making it a right-Hilbert C^*-module that is self dual). The left action of M on E is given by a unital, normal, *-homomorphism φ of M into the (W^*-) algebra of all bounded adjointable operators $\mathcal{L}(E)$ on E.

Examples

- (Basic Example) $M = \mathbb{C}$, $E = \mathbb{C}^d$, $d \geq 1$.
- $G = (G^0, G^1, r, s)$- a finite directed graph. $M = \ell^\infty(G^0)$, $E = \ell^\infty(G^1)$, $a\xi b(e) = a(r(e))\xi(e)b(s(e))$, $a, b \in M$,$\xi \in E$ $\langle \xi, \eta\rangle(v) = \sum_{s(e)=v} \overline{\xi(e)}\eta(e)$, $\xi, \eta \in E$.
- M- arbitrary , $\alpha : M \to M$ a normal unital, endomorphism. $E = M$ with right action by multiplication, left action by $\varphi = \alpha$ and inner product $\langle \xi, \eta\rangle := \xi^*\eta$. Denote it ${}_\alpha M$.
- Φ is a normal, contractive, CP map on M. $E = M \otimes_\Phi M$ is the completion of $M \otimes M$ with $\langle a \otimes b, c \otimes d\rangle = b^*\Phi(a^*c)d$ and $c(a \otimes b)d = ca \otimes bd$.

Note: If σ is a representation of M on H, $E \otimes_\sigma H$ is a Hilbert space with $\langle \xi_1 \otimes h_1, \xi_2 \otimes h_2\rangle = \langle h_1, \sigma(\langle \xi_1, \xi_2\rangle_E)h_2\rangle_H$.

Similarly, given two correspondences E and F over M, we can form the (internal) tensor product $E \otimes F$ by setting

$$\langle e_1 \otimes f_1, e_2 \otimes f_2 \rangle = \langle f_1, \varphi(\langle e_1, e_2 \rangle_E) f_2 \rangle_F$$

$$\varphi_{E \otimes F}(a)(e \otimes f)b = \varphi_E(a)e \otimes fb$$

and applying an appropriate completion.
In particular we get "tensor powers" $E^{\otimes k}$.

Also, given a sequence $\{E_k\}$ of correspondences over M, the direct sum $E_1 \oplus E_2 \oplus E_3 \oplus \cdots$ is also a correspondence (after an appropriate completion).

For a correspondence E over M we define the Fock correspondence

$$\mathcal{F}(E) := M \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \cdots$$

For every $a \in M$ define the operator $\varphi_\infty(a)$ on $\mathcal{F}(E)$ by

$$\varphi_\infty(a)(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = (\varphi(a)\xi_1) \otimes \xi_2 \otimes \cdots \otimes \xi_n$$

and $\varphi_\infty(a)b = ab$.
For $\xi \in E$, define the “shift” (or “creation”) operator T_ξ by

$$T_\xi(\xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \xi_2 \otimes \cdots \otimes \xi_n.$$

and $T_\xi b = \xi b$. So that T_ξ maps $E^{\otimes k}$ into $E^{\otimes(k+1)}$.

Definition

(1) The norm-closed algebra generated by $\varphi_\infty(M)$ and $\{T_\xi : \xi \in E\}$ will be called the **tensor algebra** of E and denoted $\mathcal{T}_+(E)$.

(2) The ultra-weak closure of $\mathcal{T}_+(E)$ will be called the **Hardy algebra** of E and denoted $H^\infty(E)$.

Examples

1. If $M = E = \mathbb{C}$, $\mathcal{F}(E) = \ell^2$, $\mathcal{T}_+(E) = A(\mathbb{D})$ and $H^\infty(E) = H^\infty(\mathbb{D})$.
2. If $M = \mathbb{C}$ and $E = \mathbb{C}^d$ then $\mathcal{F}(E) = \ell^2(\mathbb{F}_d^+)$, $\mathcal{T}_+(E)$ is Popescu's $\mathcal{A}_d$ and $H^\infty(E)$ is F_d^∞ (Popescu) or $\mathcal{L}_d$ (Davidson-Pitts). These algebras are generated by d shifts $\{S_i\}$, each S_i is an isometry and $\sum S_i S_i^* \leq I$.

Representations

Theorem

Every completely contractive representation of $\mathcal{T}_+(E)$ on H is given by a pair $(\sigma, \mathfrak{z})$ where

1. *σ is a normal representation of M on $H = H_\sigma$. ($\sigma \in NRep(M)$)*
2. *$\mathfrak{z} : E \otimes_\sigma H \to H$ is a contraction that satisfies*

$$\mathfrak{z}(\varphi(\cdot) \otimes I_H) = \sigma(\cdot)\mathfrak{z}.$$

We write $\sigma \times \mathfrak{z}$ for the representation and we have $(\sigma \times \mathfrak{z})(\varphi_\infty(a)) = \sigma(a)$ and $(\sigma \times \mathfrak{z})(T_\xi)h = \mathfrak{z}(\xi \otimes h)$ for $a \in M$, $\xi \in E$ and $h \in H$.

Write $\mathcal{I}(\varphi \otimes I, \sigma)$ for the intertwining space and $\mathbb{D}(0,1,\sigma)$ for the open unit ball there. Thus the c.c. representations of the tensor algebra are parametrized by the family $\{\overline{\mathbb{D}(0,1,\sigma)}\}_{\sigma \in NRep(M)}$.

Examples

(1) $M = E = \mathbb{C}$. So $\mathcal{T}_+(E) = A(\mathbb{D})$, σ is the trivial representation on H, $E \otimes H = H$ and $\mathbb{D}(0,1,\sigma)$ is the (open) unit ball in $B(H_\sigma)$.

(2) $M = \mathbb{C}$, $E = \mathbb{C}^d$. $\mathcal{T}_+(E) = \mathcal{A}_d$ (Popescu's algebra) and $\mathbb{D}(0,1,\sigma)$ is the (open) unit ball in $B(\mathbb{C}^d \otimes H, H)$. Thus the c.c. representations are parameterized by row contractions $(T_1, \ldots, T_d)$.

(3) M general, $E =_\alpha M$ for an automorphism α.
$\mathcal{T}_+(E) =$ the analytic crossed product.
The intertwining space can be identified with $\{X \in B(H) : \sigma(\alpha(T))X = X\sigma(T), T \in B(H)\}$ and the c.c. representations are $\sigma \times \mathfrak{z}$ where $\mathfrak{z}$ is a contraction there.

Representations of $H^\infty(E)$

The representations of $H^\infty(E)$ are given by the representations of $\mathcal{T}_+(E)$ that extend to an ultraweakly continuous representations of $H^\infty(E)$.

For a given σ, we write $\mathcal{AC}(\sigma)$ for the set of all $\mathfrak{z} \in \overline{\mathbb{D}(0,1,\sigma)}$ such that $\sigma \times \mathfrak{z}$ is a representation of $H^\infty(E)$.

We have

Theorem

$$\mathbb{D}(0,1,\sigma) \subseteq \mathcal{AC}(\sigma) \subseteq \overline{\mathbb{D}(0,1,\sigma)}.$$

Example

When $M = E = \mathbb{C}$, $H^\infty(E) = H^\infty(\mathbb{D})$ and $\mathcal{AC}(\sigma)$ is the set of all contractions in $B(H_\sigma)$ that have an H^∞-functional calculus.

Example

Induced representations: Fix a normal representation π of M on K, let $H = \mathcal{F}(E) \otimes_\pi K$ and define the representation of $H^\infty(E)$ on H by $X \mapsto X \otimes I_K$.
It is $\sigma \times \mathfrak{z}$ for $\sigma(a) = \varphi_\infty(a) \otimes I_K$ and $\mathfrak{z}(\xi \otimes h) = (T_\xi \otimes I_K)h$.
Note that $||\mathfrak{z}|| = 1$ and $\mathfrak{z} \in \mathcal{AC}(\sigma)$.

When π is faithful of infinite multiplicity we write $\sigma_0 \times \mathfrak{s}_0$ for the induced representation. It is essentially independent of π and is a universal generator in the following sense.

Universal induced representation

Theorem

Let $\sigma \times \mathfrak{z}$ be a c.c. representation of $\mathcal{T}_+(E)$ on H. Then the following are equivalent.

(1) *The representation $\sigma \times \mathfrak{z}$ extends to a c.c. ultra weakly continuous representation of $H^\infty(E)$ (that is, $\mathfrak{z} \in \mathcal{AC}(\sigma)$).*

(2) $H = \bigvee\{Ran(C) : C \in \mathcal{I}(\sigma_0 \times \mathfrak{s}_0, \sigma \times \mathfrak{z})\}$.

Here $\mathcal{I}(\sigma_0 \times \mathfrak{s}_0, \sigma \times \mathfrak{z})\}$ is the space of all maps from H_{σ_0} to H_σ that intertwine the representations $\sigma_0 \times \mathfrak{s}_0$ and $\sigma \times \mathfrak{z}$.

Partial results: Douglas (69), Davidson-Li-Pitts (05).

The families of functions

Given $F \in H^\infty(E)$, we define a family $\{\widehat{F}_\sigma\}_{\sigma \in NRep(M)}$ of (operator valued) functions.
Each function $\widehat{F}_\sigma$ is defined on $\mathcal{AC}(\sigma)$ (or on $\mathbb{D}(0,1,\sigma)$) and takes values in $B(H_\sigma)$:

$$\widehat{F}_\sigma(\mathfrak{z}) = (\sigma \times \mathfrak{z})(F).$$

Here $NRep(M)$ is the set of all normal representations of M.
Note that the family of domains (either $\{\mathcal{AC}(\sigma)\}$ or $\{\mathbb{D}(0,1,\sigma)\}$) is a matricial family in the following sense.

Definition

A family of sets $\{\mathcal{U}(\sigma)\}_{\sigma \in NRep(M)}$, with $\mathcal{U}(\sigma) \subseteq \mathcal{I}(\varphi \otimes I, \sigma)$, satisfying $\mathcal{U}(\sigma) \oplus \mathcal{U}(\tau) \subseteq \mathcal{U}(\sigma \oplus \tau)$ is called a **matricial family of sets**.

Definition

Suppose $\{\mathcal{U}(\sigma)\}_{\sigma\in NRep(M)}$ is a matricial family of sets and suppose that for each $\sigma \in NRep(M)$, $f_\sigma : \mathcal{U}(\sigma) \to B(H_\sigma)$ is a function. We say that $f := \{f_\sigma\}_{\sigma\in NRep(M)}$ is a **matricial family of functions** in case for every $\mathfrak{z} \in \mathcal{U}(\sigma)$, every $\mathfrak{w} \in \mathcal{U}(\tau)$ and every $C \in \mathcal{I}(\sigma \times \mathfrak{z}, \tau \times \mathfrak{w})$, we have

$$Cf_\sigma(\mathfrak{z}) = f_\tau(\mathfrak{w})C \tag{1}$$

Theorem

For every $F \in H^\infty(E)$, the family $\{\widehat{F}_\sigma\}$ is is a matricial family (on $\{\mathcal{AC}(\sigma)\}$).
Conversely, if $f = \{f_\sigma\}_{\sigma\in NRep(M)}$ is a matricial family of functions, with f_σ defined on $\mathcal{AC}(\sigma)$ and mapping to $B(H_\sigma)$, then there is an $F \in H^\infty(E)$ such that f is the **Berezin transform** *of F, i.e., $f_\sigma = \widehat{F}_\sigma$ for every σ.*

Notation: For $\mathfrak{z} \in \mathcal{I}(\varphi \otimes I, \sigma)$ and $k \geq 1$,
$\mathcal{Z}_k(\mathfrak{z}) = \mathfrak{z}(I_E \otimes \mathfrak{z}) \cdots (I_{E^{\otimes k-1}} \otimes \mathfrak{z}) \in \mathcal{I}(\varphi_{E^{\otimes k}} \otimes I, \sigma)$.
For a sequence $\theta = \{\theta_k\}$, with $\theta_k \in E^{\otimes k}$,

$$L_{\theta_k} : H \to E^{\otimes k} \otimes H, \quad L_{\theta_k} h = \theta_k \otimes h$$

and $R(\theta) = (\limsup_{k\to\infty} \|\theta_k\|^{\frac{1}{k}})^{-1}$. (Popescu)

Theorem

If $f = \{f_\sigma\}_{\sigma \in NRep(M)}$ is a family of functions, with f_σ mapping $\mathbb{D}(0,1,\sigma)$ to $B(H_\sigma)$, then f is a matricial family of functions if and only if there is a formal tensor series θ with $R(\theta) \geq 1$ such that f is the family of **tensorial power series** *determined by θ; that is,*

$$f_\sigma(\mathfrak{z}) = \sum_{k\geq 0} \mathcal{Z}_k(\mathfrak{z}) L_{\theta_k}.$$

Moreover, $f = \widehat{F}$ for some $F \in H^\infty(E)$ if and only if

$$\sup\{\|f_\sigma(\mathfrak{z})\| \mid \sigma \in NRep(M),\ \mathfrak{z} \in \mathbb{D}(0,1,\sigma)\} < \infty. \qquad (2)$$

Function theory without a generator

Now we fix an additive subcategory Σ of $NRep(M)$ that do not necessarily contain a special generator. Then

Theorem

Suppose that $f = \{f_\sigma\}_{\sigma\in\Sigma}$ *is a matricial family of functions defined on* $\{\mathbb{D}(0,1,\sigma)\}$ *that is locally uniformly bounded in the sense that for each* $r < 1$, $\sup_{\sigma\in\Sigma}\sup_{\mathfrak{z}\in\mathbb{D}(0,r,\sigma)}\|f_\sigma(\mathfrak{z})\| < \infty$. *Then:*

1. *Each* f_σ *is Frechet analytic on* $\mathbb{D}(0,1,\sigma)$ *and*

$$f_\sigma(\mathfrak{z}) = \sum_{n=0}^{\infty}\frac{1}{n!}D^n f_\sigma(0)(\mathfrak{z}).$$

2. *If the subcategory is full and if each* $\sigma\in\Sigma$ *is faithful, then there is* $\theta = \{\theta_k\}$ *with* $R(\theta)\geq 1$ *and*

$$f_\sigma(\mathfrak{z}) = \sum_{k\geq 0}\mathcal{Z}_k(\mathfrak{z})L_{\theta_k}$$

Now we discuss another expansion: the Taylor-Taylor series. We first need the following.

Theorem

Let $f = \{f_\sigma\}_{\sigma\in\Sigma}$ *be a matricial family of functions defined on a matricial family* $\{\mathcal{U}(\sigma)\}_{\sigma\in\Sigma}$ *where* Σ *is an additive subcategory of* $NRep(M)$*. Suppose* $\sigma, \tau \in \Sigma$, $\mathfrak{z} \in \mathcal{U}(\sigma)$, $\mathfrak{w} \in \mathcal{U}(\sigma)$ *and* $\mathfrak{u} \in \mathcal{I}(\varphi \otimes_\tau I, \sigma)$ *are such that* $\begin{pmatrix} \mathfrak{z} & \mathfrak{u} \\ 0 & \mathfrak{w} \end{pmatrix} \in \mathcal{U}(\sigma \oplus \tau)$*. Then there is an operator* $\Delta f_{\sigma,\tau}(\mathfrak{z}, \mathfrak{w})(\mathfrak{u}) \in B(H_\tau, H_\sigma)$ *such that*

$$f_{\sigma\oplus\tau}(\begin{pmatrix} \mathfrak{z} & \mathfrak{u} \\ 0 & \mathfrak{w} \end{pmatrix}) = \begin{pmatrix} f_\sigma(\mathfrak{z}) & \Delta f_{\sigma,\tau}(\mathfrak{z}, \mathfrak{w})(\mathfrak{u}) \\ 0 & f_\tau(\mathfrak{w}) \end{pmatrix}.$$

Also, the map $\mathfrak{u} \mapsto \Delta f_{\sigma,\tau}(\mathfrak{z}, \mathfrak{w})(\mathfrak{u})$ *is linear.*

Similarly, we write $\Delta^n f_{\sigma_0,\sigma_1,\cdots,\sigma_n}(\mathfrak{z}_0,\ldots,\mathfrak{z}_n)(\mathfrak{u}_1,\ldots,\mathfrak{u}_n)$ for the operator on the top-right corner of the matrix obtained by

$$f_{\sigma_0\oplus\sigma_1\oplus\cdots\oplus\sigma_n}\left(\begin{pmatrix} \mathfrak{z}_0 & \mathfrak{u}_1 & 0 & \cdots & 0 \\ 0 & \mathfrak{z}_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \mathfrak{z}_{n-1} & \mathfrak{u}_n \\ 0 & \cdots & \cdots & 0 & \mathfrak{z}_n \end{pmatrix}\right) \tag{3}$$

This map is multilinear in $\mathfrak{u}_1,\ldots,\mathfrak{u}_n$.

Definition

The function $\Delta^n f_{\sigma_0,\sigma_1,\cdots,\sigma_n}(\mathfrak{z}_0,\ldots,\mathfrak{z}_n)$ of $\mathfrak{u}_1$, $\mathfrak{u}_2$, $\cdots$, $\mathfrak{u}_n$ defined above will be called the *n^{th}-order Taylor difference operator* determined by $\mathfrak{z}_0$, $\mathfrak{z}_1$,$\cdots$,$\mathfrak{z}_n$. If $\mathfrak{z}_0 = \mathfrak{z}_1 = \cdots = \mathfrak{z}_n = \mathfrak{z}$, we call $\Delta^n f_{\sigma,\sigma,\cdots,\sigma}(\mathfrak{z},\mathfrak{z},\cdots,\mathfrak{z}) := \Delta^n f_\sigma(\mathfrak{z})$ the *n^{th}-order Taylor derivative* of f_σ at $\mathfrak{z}$.

Theorem (T-T Series)

Let $f = \{f_\sigma\}_{\sigma\in\Sigma}$ *be a matricial family of functions defined on a matricial disc* $\mathbb{D}(0,r)$ $(= \{\mathbb{D}(0,r,\sigma)\}_\sigma)$ *and suppose that* f *is locally uniformly bounded. Then:*

1. *Each* f_σ *is Frechet differentiable in* $\mathfrak{z}$, $\mathfrak{z}\in\mathbb{D}(0,r,\sigma)$, *and*

   $$f'_\sigma(\mathfrak{z})(\mathfrak{w}) = \Delta f(\mathfrak{z})(\mathfrak{w}).$$

2. $$D^k f_\sigma(0)(\mathfrak{w}) = k!\Delta^k f_\sigma(0)(\mathfrak{w}).$$

3. *Each* f_σ *may be expanded on* $\mathbb{D}(0,r,\sigma)$ *as*

   $$f_\sigma(\mathfrak{z}) = \sum_{k=0}^{\infty} \Delta^k f_\sigma(0)(\mathfrak{z},\ldots,\mathfrak{z}), \tag{4}$$

   where the series converges absolutely and uniformly on every disc $\mathbb{D}(0,r_0,\sigma)$ *with* $r_0 < r$.

Matricial family of maps

Theorem

Suppose that E and F are two W^*-correspondences over M and that $f = \{f_\sigma\}_\sigma$ is a family of maps, with $f_\sigma : \mathcal{AC}(\sigma, E) \to \mathcal{AC}(\sigma, F)$. Then f is a matricial family of maps (that is, preserves intertwiners) if and only if there is an ultraweakly continuous homomorphism $\alpha : H^\infty(F) \to H^\infty(E)$ such that for every $\mathfrak{z} \in \mathcal{AC}(\sigma, E)$ and every $Y \in H^\infty(F)$,

$$\widehat{\alpha(Y)}(\mathfrak{z}) = \widehat{Y}(f_\sigma(\mathfrak{z})). \tag{5}$$

Given two correspondences E, F over M, we will write ${}_M\mathcal{L}_M(E, F)$ for the maps in $\mathcal{L}(E, F)$ that are bimodule maps. That is, $T \in \mathcal{L}(E, F)$ lies in ${}_M\mathcal{L}_M(E, F)$ if and only if $T(\varphi_E(a)\xi b) = \varphi_F(a)T(\xi)b$, for all $a, b \in M$.

Theorem

Let E and F be two W^*-correspondences over the same W^*-algebra, M, and suppose Σ is a full additive subcategory of $NRep(M)$ whose objects are all faithful representations of M. If $f = \{f_\sigma\}_{\sigma\in\Sigma}$ is a matricial family of maps, mapping a disc $\mathbb{D}(0, r, \sigma, E)$ to a disc $\mathbb{D}(0, R, \sigma, F)$, then there is a uniquely defined sequence of maps $\{\mathfrak{D}^k f\}_{k=0}^{\infty}$, where for each k, $\mathfrak{D}^k f$ lies in ${}_M\mathcal{L}_M(F, E^{\otimes k})$, such that for every $\mathfrak{z} \in \mathbb{D}(0, r, \sigma, E)$,

$$f_\sigma(\mathfrak{z}) = f_\sigma(0) + \sum_{k\geq 1} \mathcal{Z}_k(\mathfrak{z})(\mathfrak{D}^k f \otimes I_{H_\sigma}). \quad (6)$$

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Thank You !