# Properties of derivations on some convolution algebras

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# Overview

1. Introduction
   (Weakly) compact derivations on $L^1(\omega)$
2. Derivations on $L^1[0,1)$
   (Weakly) compact derivations on $L^1[0,1)$
3. Derivations on $L^1_{\text{loc}}$
   (Weakly) compact derivations on $L^1_{\text{loc}}$
   (Weakly) Montel derivations on $L^1_{\text{loc}}$
4. Derivations on $A(\omega)$ – briefly
5. Weak-star continuity

# Motivation

- **Definition** A *derivation* on an algebra: a linear map $D : A \to A$ with

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- [Despić, Ghahramani and Grabiner, 1995]:
  Characterisation of (weakly) compact derivations on the weighted convolution algebra

$$L^1(\omega) = \{f : \mathbb{R}^+ \to \mathbb{C} \; : \; \|f\| = \int_0^\infty |f(t)|\omega(t)\,dt < \infty\}$$

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- Characterisation of (weakly) compact derivations from various Banach algebras to their dual spaces:

| | |
|---|---|
| [Choi and Heath, 2010]: | $l^1(\mathbb{Z}^+)$. |
| [Choi and Heath, 2011]: | The disc algebra. |
| [Pedersen, 2013]: | $L^1(\omega)$. |

# Derivations on $L^1(\omega)$

- For $f \in L^1(\omega)$ let
$$(Xf)(t) = tf(t) \qquad (t \in \mathbb{R}^+).$$
- For a measure $\mu$ on $\mathbb{R}^+$ and $f \in L^1(\omega)$ let
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$$D_\mu f = Xf * \mu.$$
- [Ghahramani, 1980]
  - There is a non-zero derivation on $L^1(\omega)$ iff $\exists b > 0$ such that
$$\sup_{t \in \mathbb{R}^+} \frac{t\omega(t+b)}{\omega(t)} < \infty.$$
  - The derivations on $L^1(\omega)$ are exactly $D_\mu$ where $\mu$ satisfies
$$\sup_{t \in \mathbb{R}^+} \left\{ \frac{t}{\omega(t)} \int_{\mathbb{R}^+} \omega(t+s)\, d|\mu|(s) \right\} < \infty.$$

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$$\sup_{t \in \mathbb{R}^+} \left\{ \frac{t}{\omega(t)} \int_{\mathbb{R}^+} \omega(t+s)\, d|\mu|(s) \right\} < \infty.$$
- $\overline{D}_\mu \nu = X\nu * \mu$ $(\nu \in M(\omega))$ extends $D_\mu$ to a derivation on $M(\omega)$.

# (Weakly) compact derivations on $L^1(\omega)$

**Theorem** [Despić, Ghahramani and Grabiner, 1995]

Let $D_\mu$ be a derivation on $L^1(\omega)$. The following are equivalent:

(a) $D_\mu$ is a compact derivation on $L^1(\omega)$.

(b) $D_\mu$ is a weakly compact derivation on $L^1(\omega)$.

(c) $\mu$ is absolutely continuous and

$$\frac{t}{\omega(t)} \int_{\mathbb{R}+} \omega(t+s)\, d|\mu|(s) \to 0 \text{ as } t \to \infty.$$

(d) $\overline{D}_\mu$ is a compact derivation on $M(\omega)$.

(e) $\overline{D}_\mu$ is a weakly compact derivation on $M(\omega)$.

# Oversigt

# Derivations on $L^1[0,1)$

- For $f \in L^1[0,1)$ let $(Xf)(t) = tf(t)$ $(t \in [0,1))$.
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# (Weakly) compact derivations on $L^1[0,1)$

The result of Despić, Ghahramani and Grabiner can be adapted to $L^1[0,1)$:

**Theorem**

Let $D_\mu$ be a derivation on $L^1[0,1)$. The following are equivalent:

(a) $D_\mu$ is a compact derivation on $L^1[0,1)$.

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(d) $\overline{D}_\mu$ is a compact derivation on $M[0,1)$.

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# Oversigt

# The algebra $L^1_{\text{loc}}$

- Let $L^1_{\text{loc}}$ be the space of locally integrable functions on $\mathbb{R}^+$.

  (Let $M_{\text{loc}}$ be the space of locally finite Borel measures on $\mathbb{R}^+$.)

- For $n \in \mathbb{N}$ let

$$R_n : L^1_{\text{loc}} \to L^1[0,n) \qquad \text{and} \qquad S_n : L^1[0,n) \to L^1_{\text{loc}}$$

  be the restriction and inclusion maps.

- $L^1_{\text{loc}}$ is a Fréchet convolution algebra equipped with the seminorms $f \mapsto \|R_n f\|$ $(f \in L^1_{\text{loc}})$ for $n \in \mathbb{N}$.

  Can also be regarded as the projective limit of the spaces $L^1[0,n)$.

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# (Weakly) compact derivations on $L^1_{\mathbf{loc}}$

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**Observation** A set $E \subseteq L^1_{\mathrm{loc}}$ is (weakly) compact iff $R_n(E)$ is (weakly) compact in $L^1[0,n)$ for every $n \in \mathbb{N}$.

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## Proposition

For a continuous linear operator $T$ on $L^1_{\text{loc}}$ the following are equivalent:

(a) $T$ is weakly compact.

(b) $\exists m \in \mathbb{N}\ \forall n \in \mathbb{N}$ the operator $T_{nm} = R_n T S_m : L^1[0,m) \to L^1[0,n)$ is weakly compact and $Tf = 0$ for every $f \in L^1_{\text{loc}}$ with $f = 0$ on $[0,m)$.

(c) $\exists m \in \mathbb{N}\ \forall n \in \mathbb{N}$ the operator $T_{nm} = R_n T S_m : L^1[0,m) \to L^1[0,n)$ is weakly compact and satisfies $R_n T = T_{nm} R_m$.

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## Corollary

There are no non-zero weakly compact derivations on $L^1_{\text{loc}}$.

# Problem with (weakly) compact operators on $L^1_{\text{loc}}$

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- $\exists m \in \mathbb{N}\ \forall n \in \mathbb{N}\ \exists K_n : \|R_n Tf\| \leq K_n \|R_m f\|\ (f \in L^1_{\text{loc}})$.
  [Compare with continuity.]

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**Problem** Loosely speaking neighbourhoods in $L^1_{\text{loc}}$ are too large and (weakly) compact sets in $L^1_{\text{loc}}$ are too small.

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  [Compare with continuity.]

**Problem** Loosely speaking neighbourhoods in $L^1_{\text{loc}}$ are too large and (weakly) compact sets in $L^1_{\text{loc}}$ are too small.

**“Solution”** Balls in a Banach space are prototypes of neighbourhoods but also of bounded sets.

# (Weakly) Montel derivations on $L^1_{\text{loc}}$

**Definition** A linear operator between Fréchet spaces is called *(weakly) Montel* if it maps bounded sets to (weakly) relatively compact sets.

**Observation** A set $E \subseteq L^1_{\text{loc}}$ is bounded iff $R_n(E)$ is bounded in $L^1[0, n)$ for every $n \in \mathbb{N}$.

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## Theorem

For $\mu \in M_{\text{loc}}$ the following conditions are equivalent:

(a) $D_\mu$ is a Montel derivation on $L^1_{\text{loc}}$.

(b) $D_\mu$ is a weakly Montel derivation on $L^1_{\text{loc}}$.

(c) $\mu$ is absolutely continuous.

(d) $\overline{D}_\mu$ is a Montel derivation on $M_{\text{loc}}$.

(e) $\overline{D}_\mu$ is a weakly Montel derivation on $M_{\text{loc}}$.

# Sketch of proof

- **(b)**$\Rightarrow$**(c)**:

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  $D_\mu S_m : L^1[0,m) \to L^1_{\text{loc}}$ weakly compact, $\forall m \in \mathbb{N} \quad \Rightarrow$

  $\widetilde{D}_{R_m\mu} = R_m D_\mu S_m$ weakly compact on $L^1[0,m), \ \forall m \in \mathbb{N} \quad \Rightarrow$

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  $\widetilde{D}_{R_m\mu} = R_m D_\mu S_m$ weakly compact on $L^1[0,m),\ \forall m \in \mathbb{N} \quad \Rightarrow$

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  $\mu$ absolutely continuous on $\mathbb{R}^+$.

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  $\widetilde{D}_{R_m\mu} = R_m D_\mu S_m$ weakly compact on $L^1[0,m)$, $\forall m \in \mathbb{N} \quad \Rightarrow$

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- **(c)⇒(d)**:

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  Since $R_m\overline{D}_\mu = R_m\overline{D}_\mu S_m R_m : M_{\text{loc}} \to M[0,m)$ we have
  $R_m(\overline{D}_\mu(B)) = (R_m\overline{D}_\mu S_m)(R_m(B))$ relatively compact in $M[0,m)$,
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  $\widetilde{D}_{R_m\mu} = R_m D_\mu S_m$ weakly compact on $L^1[0,m)$, $\forall m \in \mathbb{N} \quad \Rightarrow$

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  $\forall B$ bounded in $M_{\text{loc}}$ and $\forall m \in \mathbb{N} \quad \Rightarrow$

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  $\forall B$ bounded in $M_{\mathrm{loc}}$ and $\forall m \in \mathbb{N}$ $\Rightarrow$

  $\overline{D}_\mu(B)$ relatively compact in $M_{\mathrm{loc}},\ \forall B$ bounded in $M_{\mathrm{loc}}$ $\Rightarrow$

  $\overline{D}_\mu$ Montel on $M_{\mathrm{loc}}$.

# Oversigt

# Derivations on $A(\omega)$ – briefly

- For an increasing sequence $(\omega_n)$ of semisimple weights [plus some other conditions] let

$$A(\omega) = \bigcap_n L^1(\omega_n).$$

Then $A(\omega)$ is a Fréchet convolution algebra.

# Derivations on $A(\omega)$ – briefly

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- Same results as for $L^1_{\mathrm{loc}}$:
  - There are no non-zero weakly compact derivations on $A(\omega)$.
  - The (weakly) Montel derivations $D_\mu$ on $A(\omega)$ correspond to $\mu$ absolutely continuous.

# Oversigt

# Weak-star continuity

**Theorem**

All the extensions $\overline{D}_\mu$ on

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# Weak-star continuity

## Theorem

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**Sketch of proof**

- For $h \in$predual and $t \in \mathbb{R}^+$ (or $[0,1)$) let

$$(T_\mu h)(t) = t \int_{\mathbb{R}^+ \text{ (or } [0,1))} h(t+s)\, d\mu(s).$$

- Show that $T_\mu$(predual) $\subseteq$ predual.
- Show that $\overline{D}_\mu = T_\mu^*$.