# Corona Theorems for Multiplier Algebras on the Unit Ball

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Banach Algebras 2013
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# Where Did the Name Come From?

## The Corona Problem for $H^\infty(\mathbb{D})$

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To each $z \in \mathbb{D}$ we can associate a multiplicative linear functional on $H^\infty(\mathbb{D})$:

$$\varphi_z(f) \equiv f(z) \quad \text{(point evaluation at } z\text{)}.$$

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The proceeding discussion then shows that $\mathbb{D} \subset \mathcal{M}_{H^\infty(\mathbb{D})}$.

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If $f_1, \ldots, f_N \in H^\infty(\mathbb{D})$ and if

$$\max_{1 \leq j \leq N} |f_j(z)| \geq \delta > 0 \quad \forall z \in \mathbb{D}$$

then there exists $g_1, \ldots, g_N \in H^\infty(\mathbb{D})$ such that

$$1 = \sum_{j=1}^{N} f_j(z) g_j(z).$$

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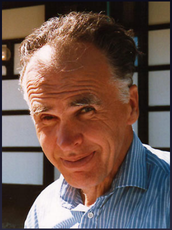

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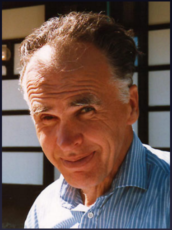

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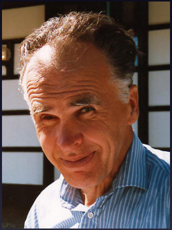

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**Theorem (Carleson's Corona Theorem)**

*Let* $\{f_j\}_{j=1}^N \in H^\infty(\mathbb{D})$ *satisfy*

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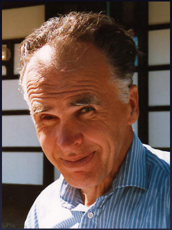

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*Then there are functions $\{g_j\}_{j=1}^N$ in $H^\infty(\mathbb{D})$ with*

$$\sum_{j=1}^N f_j(z)\, g_j(z) = 1 \; \forall z \in \mathbb{D} \text{ and } \|g_j\|_{H^\infty(\mathbb{D})} \leq C_{\delta,N}.$$

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**Conjecture (Corona Problem for the Polydisc or Ball)**

*When $n \geq 2$ and $X$ is either the $\mathbb{B}_n$ or $\mathbb{D}^n$ the $H^\infty(X)$-Corona Problem is true.*

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*Given* $f_1, \ldots, f_N \in M_{\mathcal{H}}$ *satisfying*

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$$\begin{aligned} \|l_1\|_{\mathcal{H}}^2 + \cdots + \|l_N\|_{\mathcal{H}}^2 &\leq C\|h\|_{\mathcal{H}}^2, \\ l_1(z) f_1(z) + \cdots + l_N(z) f_N(z) &= h(z) \quad \forall z \in X? \end{aligned}$$

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Where $l_j \equiv g_j h$ and $\|l_j\|_{\mathcal{H}} \leq \|g_j\|_{M_{\mathcal{H}}} \|h\|_{\mathcal{H}}$.

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## Complete Nevanlinna-Pick Kernels

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A kernel $k$ is a *complete Nevanlinna-Pick* (CNP) kernel for $\mathcal{H}$ if for any finite set of $n$ distinct points $\{x_1, \ldots, x_n\} \subset X$ the matrix

$$\left(1 - \frac{k_{x_i}(x_n)k_{x_n}(x_j)}{k_{x_i}(x_j)k_{x_n}(x_n)}\right)_{i,j=1}^{n-1}$$

is positive semi-definite.

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$$\left(1 - \frac{k_{x_i}(x_n)k_{x_n}(x_j)}{k_{x_i}(x_j)k_{x_n}(x_n)}\right)_{i,j=1}^{n-1}$$

is positive semi-definite.

### Example (Kernels on $\mathbb{D}$)

$k_z(w) = (1 - \overline{z}w)^{-1}$ *is CNP. But,* $k_z(w) = (1 - \overline{z}w)^{-2}$ *is not.*

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## Toeplitz Corona Theorem

### Theorem (Toeplitz Corona Theorem, (Agler and McCarthy))

*Let $\mathcal{H}$ be a Hilbert function space on an open set $X$ in $\mathbb{C}^n$ with an irreducible complete Nevanlinna-Pick kernel. Let $\epsilon > 0$ and let $f_1, \ldots, f_N \in M_{\mathcal{H}}$. Then the following are equivalent:*

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(i) *There exists $g_1, \ldots, g_N \in M_{\mathcal{H}}$ such that $\sum_{j=1}^N f_j g_j = 1$ and $\sum_{j=1}^N \|g_j\|_{M_{\mathcal{H}}} \leq \frac{1}{\epsilon}$;*

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(ii) *For any $h \in \mathcal{H}$, there exists $l_1, \ldots, l_N \in \mathcal{H}$ such that $h = \sum_{j=1}^N l_j f_j$ and $\sum_{j=1}^N \|l_j\|_{\mathcal{H}}^2 \leq \frac{1}{\epsilon^2} \|h\|_{\mathcal{H}}^2$.*

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**Moral:** If the Hilbert space has a reproducing kernel with enough structure, then the Corona Problem and the Baby Corona Problem are the same question.

# The $\overline{\partial}$-problem and the Koszul Complex

First consider the case of two functions so that we can see the connections between this problem and the $\overline{\partial}$-problem we will study.

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Define the following functions

$$\varphi_1(z) \equiv \frac{\overline{f_1(z)}}{|f_1(z)|^2 + |f_2(z)|^2} \quad \varphi_2(z) \equiv \frac{\overline{f_2(z)}}{|f_1(z)|^2 + |f_2(z)|^2}.$$

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The hypotheses on $f_1$ and $f_2$ imply that the functions $\varphi_1$ and $\varphi_2$ are in fact bounded and smooth on $X$. Note that

$$\varphi_1(z) f_1(z) + \varphi_2(z) f_2(z) = 1 \quad \forall z \in X$$

but the functions $\varphi_1$ and $\varphi_2$ are in general *not* analytic.

# Motivating the $\overline{\partial}$-problem

Now, observe for any function $r$ we have that the functions

$$g_1 = \varphi_1 + rf_2 \quad g_2 = \varphi_2 - rf_1$$

also solve the problem

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Using these two equations and the condition that $f_1\varphi_1 + f_2\varphi_2 = 1$ gives that the function $r$ must satisfy the equation

$$\overline{\partial} r = \varphi_1\overline{\partial}\varphi_2 - \varphi_2\overline{\partial}\varphi_1.$$

# The Koszul Complex

- If $f = (f_j)_{j=1}^N$ satisfies $|f|^2 = \sum_{j=1}^N |f_j|^2 \geq 1$, let

$$\Omega_0^1 = \frac{\overline{f}}{|f|^2} = \left(\frac{\overline{f_j}}{|f|^2}\right)_{j=1}^N = \left(\Omega_0^1(j)\right)_{j=1}^N,$$

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- Then $\varphi = \Omega_0^1 h$ satisfies $f \cdot g = h$, but in general fails to be analytic.

- The Koszul complex provides a scheme when $f$ and $h$ are holomorphic for solving a sequence of $\overline{\partial}$ equations that result in a correction term $\Lambda_f \Gamma_0^2$ that when subtracted from $\varphi$ above yields an *analytic* solution to $f \cdot g = h$.

# Lifting of Forms

- The 1-tensor of $(0,1)$-forms $\overline{\partial}\Omega_0 = \left(\overline{\partial}\frac{\overline{f_j}}{|f|^2}\right)_{j=1}^{N} = \left(\overline{\partial}\Omega_0^1(j)\right)_{j=1}^{N}$ is given by

$$\overline{\partial}\Omega_0^1(j) = \overline{\partial}\frac{\overline{f_j}}{|f|^2} = \frac{1}{|f|^4}\sum_{k=1}^{N} f_k \overline{\{f_k \partial f_j - \partial f_k f_j\}}.$$

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- A key fact is that this 1-tensor of $(0,1)$-forms can be "factored" as

$$\overline{\partial}\Omega_0^1 = \Lambda_f\Omega_1^2 \equiv \left[\sum_{k=1}^{N}\Omega_1^2(j,k)\,f_k\right]_{j=1}^{N},$$

where the 2-tensor $\Omega_1^2$ of $(0,1)$-forms is given by

$$\Omega_1^2 = \left[\Omega_1^2(j,k)\right]_{j,k=1}^{N} = \left[\frac{\overline{\{f_k\partial f_j - \partial f_k f_j\}}}{|f|^4}\right]_{j,k=1}^{N}.$$

# Solving the complex ...

- We can repeat this process and by induction we have

$$\overline{\partial}\Omega_q^{q+1} = \Lambda_f \Omega_{q+1}^{q+2}, \qquad 0 \leq q \leq n,$$

where $\Omega_q^{q+1}$ is an *alternating* $(q+1)$-tensor of $(0,q)$-forms.

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- Recall that $h$ is holomorphic. When $q = n$ we have that $\Omega_n^{n+1} h$ is $\overline{\partial}$-closed since every $(0,n)$-form is $\overline{\partial}$-closed.

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- This allows us to begin solving a chain of $\overline{\partial}$ equations

$$\overline{\partial}\Gamma_{q-2}^{q} = \Omega_{q-1}^{q} h - \Lambda_f \Gamma_{q-1}^{q+1}$$

# ...using that the forms are closed

- Since $\Omega_n^{n+1}h$ is $\overline{\partial}$-closed and alternating, there is an alternating $(n+1)$-tensor $\Gamma_{n-1}^{n+1}$ of $(0, n-1)$-forms satisfying

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- Now note that the $n$-tensor $\Omega_{n-1}^{n}h - \Lambda_f \Gamma_{n-1}^{n+1}$ of $(0, n-1)$-forms is $\overline{\partial}$-closed:

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- Thus there is an alternating $n$-tensor $\Gamma_{n-2}^{n}$ of $(0, n-2)$-forms satisfying:

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- With the convention that $\Gamma_n^{n+2} \equiv 0$, induction shows that there are alternating $(q+2)$-tensors $\Gamma_q^{q+2}$ of $(0,q)$-forms for $0 \leq q \leq n$ satisfying

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- Now

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is holomorphic by the first line above, and since $\Gamma_0^2$ is antisymmetric, we compute that $\Lambda_f \Gamma_0^2 \cdot f = \Gamma_0^2(f,f) = 0$ and

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- Thus $g = (g_1, g_2, \ldots, g_N)$ is an $N$-vector of holomorphic functions satisfying $f \cdot g = h$.

# Besov-Sobolev Spaces

The space $B_2^\sigma(\mathbb{B}_n)$ is the collection of holomorphic functions $f$ on the unit ball $\mathbb{B}_n$ such that

$$\left\{\sum_{k=0}^{m-1}\left|f^{(k)}(0)\right|^2+\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m+\sigma}f^{(m)}(z)\right|^2 d\lambda_n(z)\right\}^{\frac{1}{2}}<\infty,$$

where $d\lambda_n(z)=\left(1-|z|^2\right)^{-n-1}dV(z)$ is the invariant measure on $\mathbb{B}_n$ and $m+\sigma>\frac{n}{2}$.

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- $\sigma=\frac{n}{2}$: Classical Hardy Space;
- $\sigma>\frac{n}{2}$: Bergman Spaces.

# Besov-Sobolev Spaces

The spaces $B_2^\sigma(\mathbb{B}_n)$ are examples of reproducing kernel Hilbert spaces.

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The spaces $B_2^\sigma(\mathbb{B}_n)$ are examples of reproducing kernel Hilbert spaces. Namely, for each point $\lambda \in \mathbb{B}_n$ there exists a function $k_\lambda \in B_2^\sigma(\mathbb{B}_n)$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle_{B_2^\sigma}.$$

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When $0 \leq \sigma \leq \frac{1}{2}$ then $k_\lambda$ is a complete Nevanlinna-Pick kernel.

# Baby Corona Theorem for $B_p^\sigma(\mathbb{B}_n)$

## Theorem (Ş. Costea, E. Sawyer, BDW; Analysis & PDE **4** (2011))

*Let* $0 \leq \sigma$ *and* $1 < p < \infty$. *Given* $f_1, \ldots, f_N \in M_{B_p^\sigma}(\mathbb{B}_n)$ *satisfying*

$$0 < \delta \leq \sum_{j=1}^{N} |f_j(z)|^2 \leq 1, \qquad z \in \mathbb{B}_n,$$

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$$\begin{aligned}\sum_{j=1}^{N} \|k_j\|_{B_p^\sigma(\mathbb{B}_n)}^p &\leq C_{n,\sigma,N,p,\delta} \|h\|_{B_p^\sigma(\mathbb{B}_n)}^p, \\ \sum_{j=1}^{N} k_j(z) f_j(z) &= h(z) \quad \forall z \in \mathbb{B}_n.\end{aligned}$$

# The Corona Theorem for $M_{B_2^\sigma}(\mathbb{B}_n)$

**Theorem (Ş. Costea, E. Sawyer, BDW; Analysis & PDE 4 (2011))**

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# Easy Corollaries

## Corollary

*For* $0 \leq \sigma \leq \frac{1}{2}$*, the unit ball* $\mathbb{B}_n$ *is dense in the maximal ideal space of* $M_{B_2^\sigma}(\mathbb{B}_n)$*.*

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## Theorem (Ş. Costea, E. Sawyer, BDW)

*For $1 < p < \infty$ and $0 \leq \sigma < \infty$ the Taylor spectrum for the tuple $f \in \left(B_p^\sigma(\mathbb{B}_n)\right)^m$ is given by $\sigma\left(f, B_\sigma^p(\mathbb{B}_n)\right) = \overline{f(\mathbb{B}_n)}$.*

# Sketch of Proof of the Baby Corona Theorem

Given $f_1, \ldots, f_N \in M_{B_p^\sigma}(\mathbb{B}_n)$ satisfying

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- This solution is smooth and satisfies the correct estimates, but is far from analytic.
- In order to have an analytic solution we utilize the Koszul complex to produce analytic solutions $g$ such that $\sum_{j=1}^N f_j(z) g_j(z) = h(z)$.
- However, the estimates we seek are now unfortunately in doubt. We need to show that solutions to $\overline{\partial}$-problems "preserve" $B_p^\sigma(\mathbb{B}_n)$ norms.

# Estimates in $B_p^\sigma$ and $\overline{\partial}$-problems

Given a form $(0,q)$-form $\eta$ we can solve $\overline{\partial} u = \eta$ by

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$$\frac{(1 - w\overline{z})^{n-q}\left(1 - |w|^2\right)^{q-1}}{\triangle(w,z)^n}(\overline{w}_j - \overline{z}_j) \quad \forall\, 1 \leq q \leq n.$$

Here $\triangle(w,z) = |1 - w\overline{z}|^2 - \left(1 - |w|^2\right)\left(1 - |z|^2\right)$.

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  Here $\triangle(w,z) = |1 - w\overline{z}|^2 - \left(1 - |w|^2\right)\left(1 - |z|^2\right)$.
- One then needs to show that these solution operators map the Besov-Sobolev spaces $B_\sigma^p(\mathbb{B}_n)$ to themselves. This is accomplished by a couple of key facts.

# Key Properties in Proving the Estimates

- The Besov-Sobolev spaces are very "flexible" in terms of the norm that one can use. One need only take the parameter $m$ sufficiently high.

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- To show that the solution operators are bounded on $L^p(\mathbb{B}_n; dV)$ the original proof uses the Schur Test. To handle the boundedness on $B^p_\sigma(\mathbb{B}_n)$, we can also use the Schur test but this requires a little more work to handle the derivative.

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- The fact that the Corona data belongs to the multiplier algebra allows one to prove certain embedding theorems that are used to control terms in the application of the Schur test.
- Hard work (and technical estimates that we omit!) then lets you conclude that the solutions obtained by the Koszul complex have the desired estimates.

# The $H^\infty(\mathbb{B}_n)$ Corona Problem

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*Given* $f_1, \ldots, f_N \in H^\infty(\mathbb{B}_n)$ *satisfying* $0 < \delta \leq \sum_{j=1}^N |f_j(z)|^2 \leq 1$ *for all* $z \in \mathbb{B}_n$.

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This gives another proof of a famous theorem of Varopoulos.

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**Theorem (Equivalence between Corona and Baby Corona, (Amar 2003))**

*Let $\{g_j\}_{j=1}^N \subseteq H^\infty(\mathbb{B}_n)$. Then there exists $\{f_j\}_{j=1}^N \subseteq H^\infty(\mathbb{B}_n)$ with*

$$\sum_{j=1}^N f_j(z)g_j(z) = 1 \quad \forall z \in \mathbb{B}_n \quad \text{and} \quad \sum_{j=1}^N \|g_j\|_{H^\infty(\mathbb{B}_n)} \leq \frac{1}{\delta}$$

*if and only if*

$$\mathcal{M}_g^\mu(\mathcal{M}_g^\mu)^* \geq \delta^2 I_\mu$$

*for all probability measures $\mu$ on $\partial\mathbb{B}_n$.*

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This is a great theorem since it suggests how to attack the Corona Problem for $H^\infty(\mathbb{B}_n)$. But, the difficulty is that one must solve the Baby Corona Problem for **every** probability measure on $\partial\mathbb{B}_n$.

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**Theorem (Trent, BDW; Complex Anal.Oper. Theory 3 (2009))**

*Assume that* $\mathcal{M}_g^H \mathcal{M}_g^{H*} \geq \delta^2 I_H$ *for all* $H \in \mathcal{W}$. *Then there exists a* $f_1, \ldots, f_N \in H^\infty(\mathbb{B}_n)$, *so that*

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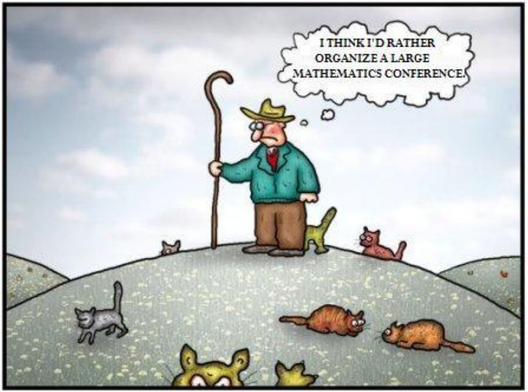
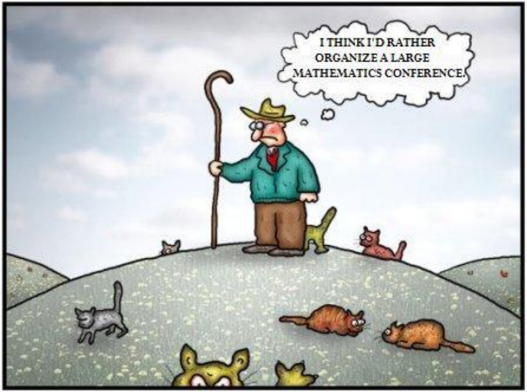
This reduces the $H^\infty(\mathbb{B}_n)$ Corona Problem to a certain "weighted" Baby Corona Problem.




The daydreams of cat herders

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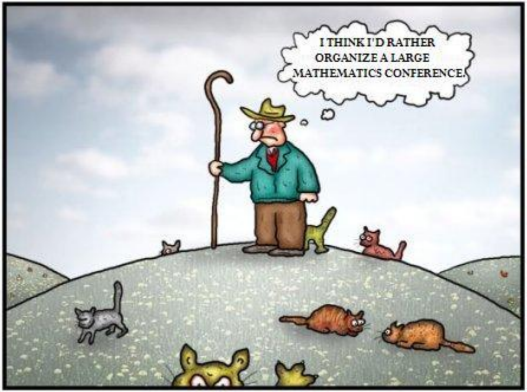

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Thanks to Lyudmyla, Maria, and Sandra for the Invitation and Organizing the Meeting!

Thank You!

Tack Så Mycket!