Toeplitz projections and essential commutants

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Unit disc

$H = H^2(\mathbb{T}), M_f \in B(L^2(\mathbb{T}))$ multiplication with $f \in L^\infty(\mathbb{T})$

$$T_f = PM_f|H^2(\mathbb{T}) \text{ Toeplitz operator with symbol } f$$

Essential commutant of $\mathcal{T} \subset B(H)$

$$\mathcal{T}^{ec} = \{S \in B(H); ST - TS \in \mathcal{K}(H) \text{ for all } T \in \mathcal{T}\}$$

Calculate $\mathcal{T}^{ec}$ for all Toeplitz operators

$$\mathcal{T} = \{T_f; f \in L^\infty(\mathbb{T})\}$$

and the class of analytic Toeplitz operators

$$\mathcal{T}_a = \{T_f; f \in H^\infty(\mathbb{T})\}$$

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Sarason/Douglas 72 : For $f \in L^\infty(\mathbb{T})$ we have

$$T_f \in \mathcal{T}_a^{ec} \Leftrightarrow f \in H^\infty(\mathbb{T}) + C(\mathbb{T})$$

Davidson 77:

$$\mathcal{T}_a^{ec} = \{T_f + K; f \in H^\infty(\mathbb{T}) + C(\mathbb{T}),\ K \in \mathcal{K}\}$$

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Unit ball: Sarason algebra

$H = H^2(S,\sigma)$ Hardy space on the unit sphere $S = \partial\mathbb{B}_n$, σ surface measure on S

Davie-Jewell 77 : For $n > 1$

$$H^\infty(S) + C(S) \subsetneq \mathcal{S} = \{f \in L^\infty(\sigma);\ H_f \text{ compact}\} \subset L^\infty(\sigma) \text{ closed subalgebras}$$

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Toeplitz operators

Let $T \in B(H)^n$ be subnormal with minimal normal extension $U \in B(\widehat{H})^n$ and

$$\Psi_U : L^\infty(\mu) \xrightarrow{\sim} W^*(U) \quad \text{functional calculus of } U \quad \text{(vNA isomorphism)}$$

Define Toeplitz and Hankel operators with symbol $f \in L^\infty(\mu)$ by

$$T_f = P_H \Psi_U(f)|H$$

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The restriction algebra

$$\mathcal{R}_T = \{f \in L^\infty(\mu);\ \Psi_U(f)H \subset H\} \subset L^\infty(\mu)$$

is a w^*-closed subalgebra.

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A-Isometries

Let $T \in B(H)^n$ be subnormal with minimal normal extension $U \in B(\widehat{H})^n$ and let

$K \subset \mathbb{C}^n$ be compact, $\quad \mathbb{C}[z]|K \subset A \subset C(K)$ a closed subalgebra.

T is called an A-isometry if $\sigma(U) \subset \partial_A$ (Shilov boundary) and

$$H^\infty_A(\mu) := \overline{A}^{w^*} \subset \mathcal{R}_T = \{f \in L^\infty(\mu);\ \Psi_U(f)H \subset H\}$$

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Examples of A-isometries

- $A = A(\mathbb{B}_n) = \{f \in C(\overline{\mathbb{B}}_n)\,;\, f|\mathbb{B}_n \in \mathcal{O}(\mathbb{B}_n)\}$. Then $\partial_A = \partial\mathbb{B}_n$

  Athavale 90 : $T \in B(H)^n$ $A-$isometry $\Leftrightarrow \sum_{i=1}^{n} T_i^* T_i = 1_H$ (spherical isometries)

- $A = A(\mathbb{D}^n)$. Then $\partial_A = \mathbb{T}^n$

  Ito 58 : $T \in B(H)^n$ $A-$isometry $\Leftrightarrow T_i^* T_i = 1_H$ $(1 \leq i \leq n)$

- $D \subset \mathbb{C}^n$ strictly pseudoconvex or symmetric, $A = A(D)$

  $T = M_z \in B(H^2(\sigma))^n,\ U = M_z \in B(L^2(\sigma))^n,\ \sigma \in M_1^+(\partial_A)$ canonical

- $A = C(K)$. Then $\partial_A = K,\quad H_A^\infty(\mu) = L^\infty(\mu)$ and

  $T \in B(H)^n$ $A-$isometry $\Leftrightarrow$ T normal with $\sigma(T) \subset K$.

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Essential commutants : Problem

Let $T \in B(H)^n$ be an A-isometry and $\mathcal{T}_a = \{T_f;\ f \in H_A^\infty(\mu)\}$.

Question: Is $\mathcal{T}_a^{ec} = \{T_f + K\,;\ H_f, K \text{ compact}\}$?

$$\text{Answer : } \supset \text{ Yes ! } \quad \subset \text{ No !}$$

Brown-Halmos 63 : On $H = H^2(\mathbb{T})$

$$\mathcal{T} = \{X \in B(H)\,;\ T_\theta^* X T_\theta = X \ \forall\ \theta \in H^\infty(\mathbb{T}) \text{ inner}\}$$

Define μ- inner functions

$$I_\mu := \{\theta \in H_A^\infty(\mu)\,;\ |\theta| = 1\ \mu - a.e.\},$$

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Toeplitz projection for T

An A-isometry $T \in B(H)^n$ is called regular if $W^*(I_\mu) = L^\infty(\mu)$

Aleksandrov 84 : All given examples of A-isometries are regular.

Arveson 75: $\exists$ positive unital projection from $B(H^2(\mathbb{T}))$ onto $\{T_f; f \in L^\infty(\mathbb{T})\}$.

Theorem (Prunaru 2007, Everard - E.)

If $\{\theta_k\} \subset I_\mu$ is w^*- dense, then $\Phi_T : B(H) \to B(H)$,

$$\Phi_T(X) = w^* - \lim_k \frac{1}{k^k} \sum_{i_1,\dots,i_k=1}^{k} (T_{\theta_1^{i_1}\dots\theta_k^{i_k}})^* X T_{\theta_1^{i_1}\dots\theta_k^{i_k}}$$

defines a completely positive unital projection onto

$$\mathrm{Im}\,\Phi_T = \{X \in B(H);\ T_\theta^* X T_\theta = X \text{ for all } \theta \in I_\mu\} = \mathcal{T}(T).$$

Is there a natural relation between abstract and concrete Toeplitz operators?

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Toeplitz projection for U

The map $\Phi_U : \; B(\widehat{H}) \to B(\widehat{H})$ defines a projection onto

$$\mathcal{T}(U) = \{X \in B(\widehat{H});\; X\Psi_U(\theta) = \Psi_U(\theta)X \text{ for all } \theta \in I_\mu\} = W^*(U)' = (U)'$$

with

$$\Phi_T(X) = P(\Phi_U(X \oplus 0))|H \qquad (X \in B(H)).$$

The minimal Stinespring dilation of $\Phi_T|C^*(\mathcal{T}(T))$ is given by

$$\pi : \; C^*(\mathcal{T}(T)) \to B(\widehat{H}),\; \pi(X) = \Phi_U(X \oplus 0).$$

Corollary

(a) $\mathcal{T}(T) = \{PX|H;\; X \in W^*(U)'\}$

(b) *For all* $f \in L^\infty(\mu)$

- $\|T_f\| = \|P_H\Psi_U(f)|H\| = \|\Psi_U(f)\| = \|f\|_{L^\infty(\mu)}$
- $f(\partial_A) \subset \sigma(T_f)$ *(Hartman-Wintner)*

Essential commutants

Theorem (Didas, E., Everard)

If $T \in B(H)^n$ is an essentially normal regular A-isometry, then

$$\mathcal{T}_a^{ec} \subset \mathcal{T}(T) + \mathcal{K}(H) .$$

Idea: Let $S \in \mathcal{T}_a^{ec}$ be given. Show that $S - \Phi_T(S) \in \mathcal{K}(H)$.

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Applications: T regular A-isometry

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T essentially normal with $W^*(U)' = W^*(U)$ (e.g. D strictly pseudoconvex), then

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Corollary (Johnson-Parrot 72)

$T \in B(H)^n$ normal $\Rightarrow$ $W^*(T)^{ec} = W^*(T)' + \mathcal{K}(H)$

Proof: Apply the main theorem with $T = U$, $A = C(\sigma(T))$ to obtain that

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Let $T \in B(H)^n$ be a regular A-isometry. Equivalent are:

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Derivations: T regular A-isometry with $\sigma_p(T) = \emptyset$

Let $B \subset \mathcal{S} = \{f \in L^\infty(\mu);\ H_f \text{ compact}\}$ be a closed subalgebra

Lemma

There is a unique short exact sequence

$$0 \longrightarrow \mathcal{K}(H) \hookrightarrow \mathcal{T}(B) + \mathcal{K}(H) \xrightarrow{\rho} B \longrightarrow 0$$

of Banach algebras with $\rho(T_f) = f$ for $f \in B$.

Theorem

There are canonical linear maps

$$\mathrm{Der}(\mathcal{T}(B) + \mathcal{K}(H))/\{\text{inner}\} \ \cong\ \mathcal{T}(B)^{ec}/\mathcal{T}(B) + \mathcal{K}(H) \ \hookleftarrow\ \mathcal{S}/B$$

Proof. $D \in \mathrm{Der}\,(\mathcal{T}(B) + \mathcal{K}) \ \Rightarrow\ D(\mathcal{K}(H)) \subset \mathcal{K}(H)$ and

$$\exists S \in B(H) \text{ with } D = [\cdot, S] \text{ on } \mathcal{T}(B) + \mathcal{K}(H)$$

$\Rightarrow D/\mathcal{K}(H) \in \mathrm{Der}(\mathcal{T}(B) + \mathcal{K}(H)/\mathcal{K}(H)) = \mathrm{Der}(B) = \{0\}$ (Singer-Wermer)

Related results and problems

Essential spectra: $f = (f_1, \ldots, f_m) \in (H^\infty(\partial D) + C(\partial D))^m$, $F = (P[f_1], \ldots, P[f_m])$

$$\Rightarrow \sigma_e(T_f) = \bigcap (\overline{F(U \cap D)};\ U \supset \partial D \, \text{open}) \text{ is connected.}$$

Same result for $f \in \mathcal{S}^m$? Can show: $\subset$ and $\sigma_e(T_f)$ is connected.

Maximal ideal space of $\mathcal{S}$:

$$\{\delta_\Lambda :\ \mathcal{S} \to \mathbb{C},\ f \mapsto F^\beta(\Lambda);\ \Lambda \in \beta(\mathbb{B}) \setminus \mathbb{B}\} \subset \Delta_{\mathcal{S}}$$

What about equality? Yes for n = 1.

Polydisc case (n > 1):

$$\mathcal{S} = H^\infty(\mathbb{T}^n) \text{ (Cotlar, Sadosky 93)}, \quad T_f \in \mathcal{T}_a^{ec} \overset{E}{\Longleftrightarrow} f \in H^\infty(\mathbb{T}^n)$$

Is $\mathcal{T}_a^{ec} = \{T_f + K; f \in H^\infty(\mathbb{T}^n), K \text{ compact}\}$?