Decomposing compositions
and
three theorems of Frostman

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Joint work with

Finite Blaschke products: Ueli Daepp, Ben Sokolowsky, Andrew Shaffer, Karl Voss (Bucknell University)

Three theorems of Frostman: John Akeroyd (University of Arkansas, Fayetteville)
Inner function: $I : \mathbb{D} \to \mathbb{D}$ analytic with radial limits of modulus 1 a.e.

**Definition**

An inner function $I$ is **indecomposable** or **prime** if whenever $I = U \circ V$ with $U$ and $V$ inner, either $U$ or $V$ is a disk automorphism.

Question: Which inner functions can be prime?

Motivation from composition operators: $C_\Phi : X \to X$ defined by $C_\Phi(f) = f(\Phi)$. 
One reason to care

Range of composition operators:

**Theorem (J. Ball, 1975; K. Stephenson 1979, (revised))**

Let $X$ be any $H^p$ space, $0 < p \leq \infty$. and let $M$ be a linear submanifold of $X$ that is closed under uniform convergence on compact subsets of $\mathbb{D}$. Then $M = C_\Phi(X)$ for some inner function $\Phi$, if and only if $M$ has the following properties:

1. $M$ contains a nonconstant function.
2. If $f, g \in M$ and $f \cdot g \in X$ (resp. $f/g \in X$), then $f \cdot g \in M$ (resp. $f/g \in M$).
3. If $f \in M$ and $I$ is the inner factor of $f$, then $I \in M$.
4. $M$ contains g.c.d. $\{B \in M : B$ inner $B(0) = 0\}$. 
First: **Finite Blaschke products**

\[ B(z) = \lambda \prod_{j=1}^{n} \frac{a_j - z}{1 - \overline{a_j}z} \text{, where } |a_j| < 1, |\lambda| = 1; \varphi_a(z) = \frac{a - z}{1 - \overline{a}z}. \]

1922-3, J. Ritt reduced to result about groups (Trans. AMS): \( F \) is a composition iff the group of \( F^{-1}(w) \) is imprimitive.

1974: Carl Cowen gave result for rational functions. (ArXiv)

The group: Associated with the set of covering transformations of the Riemann surface of the inverse of the Blaschke product; Compositions correspond to (proper) normal subgroups.

2000, JLMS Beardon, Ng simplified Ritt’s work, 2011 Tsang and Ng, Extended to finite mappings between Riemann surfaces
Basic Assumptions

$B$ has distinct zeros.

$\varphi_a(z) = (a - z)/(1 - \overline{a}z)$

$B$ is indecomposable iff $\varphi_{B(0)} \circ B$ is, so we suppose $B(0) = 0$. 

$B = C \circ D$ with $C, D$ Blaschke iff $B = (C \circ \varphi_D(0)) \circ (\varphi_D(0) \circ D)$ is. So we suppose $B(0) = C(0) = D(0) = 0$. Nice consequence: $C(z) = zC_1(z)$; $B(z) = C(D(z)) = D(z)(C_1(D(z)))$ and $D$ is a subfactor of $B$. 
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\( B(z) = C(D(z)) = D(z)(C_1(D(z))) \) and \( D \) is a subfactor of \( B \).
Algorithm 1: Try all sets of zeros \((B \text{ degree } n, B(0) = 0)\)

Is \(B(z) = C \circ D(z)\)?

\[
\text{degree}(D) = k, \text{degree}(C) = m, \text{degree}(B) = mk = n
\]

Pick subsets of size \(k\) to be the zeros of \(D\) (include 0) \(D\) is \(k - 1\) so \(D\) partitions the zeros of \(B\) into \(m\) sets of \(k\) points. You’re done.

Theorem (Algorithm 1.)

\(B = C \circ D\) with \(D\) degree \(k\) iff there is a subproduct \(D\) of \(B\) of degree \(k\) that identifies the zeros of \(B\) in \(m\) sets of \(k\) points.

But you don’t know anything about your Blaschke product. Won’t work for infinite Blaschke products.
Algorithm 2: Critical Points ($B$ degree $n$, $B(0) = 0$)

Critical point: $B'(z) = 0$; critical value $w = B(z), B'(z) = 0$.

Theorem (Heins, 1942; Zakeri, BLMS 1998)

Let $z_1, \ldots, z_d \in \mathbb{D}$. There exists a unique Blaschke $B$, degree $d + 1$, $B(0) = 0$, $B(1) = 1$, and $B'(z_j) = 0$, all $j$.

Corollary (Nehari, 1947; Zakeri)

Blaschke pdts. $B_1, B_2$ have the same critical pts. iff $B_1 = \varphi_a \circ B_2$ for some automorphism $\varphi_a$.

Remark. $B$ with distinct zeros has $2n - 2$ critical points,
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**Remark.** $B$ with distinct zeros has $2n - 2$ critical points, only $n - 1$ are in $\mathbb{D}$: $\{z_1, \ldots, z_{n-1}, 1/z_1, \ldots, 1/z_{n-1}\}$: $B$ has $\leq n - 1$ critical values in $\mathbb{D}$. 
Algorithm 2: Counting critical values

\[ B = C \circ D \implies B'(z) = C'(D(z))D'(z); \text{ } D \text{ has } k - 1 \text{ critical points, } D \text{ partitions the others into } m - 1 \text{ sets.} \]

**Theorem**

\[ B = C \circ D \text{ iff there exists a subproduct } D \text{ of } B \text{ sharing } k - 1 \text{ critical pts. with } B \text{ that partitions the others into } m - 1 \text{ sets.} \]

\[ B \] can have at most \((k - 1)\)
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\[ B \text{ can have at most } (k - 1) + (m - 1) \text{ critical values.} \]
Which one is a composition?

Note: Argument chooses the color

Figure: Blaschke products of degree 16
Which one is a composition?

Note: Argument chooses the color

Figure: Blaschke products of degree 16
Theorem (Poncelet’s porism)

Let $\mathcal{C}$ and $\mathcal{D}$ be two ellipses. If $\mathcal{C}$ is inscribed in one $n$-gon with vertices on $\mathcal{D}$, then $\mathcal{C}$ is inscribed in every $n$-gon with vertices on $\mathcal{D}$. 
Other things want to be Poncelet curves:

**Figure:** Acts like a Poncelet curve

**Definition**

$C \subset \mathbb{D}$ is a **Poncelet curve** if whenever $C$ is inscribed in one $n$-gon with vertices on $\mathbb{T}$, every $\lambda \in \mathbb{T}$ is the vertex of such an $n$-gon.
Work of Gau-Wu and Daepp, G., Voss implies

<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Every Blaschke product $B$, $B(0) = 0$ degree $n$, is associated with a unique such Poncelet curve; $B$ identifies the vertices of the $n$-gon.</td>
</tr>
</tbody>
</table>

Can we pair Poncelet curves with Blaschke products?

No: Every Blaschke product is associated with a Poncelet curve, but not every Poncelet curve is associated with a Blaschke product. Those that are will be called **B-Poncelet** curves.

**Theorem (DGSSV)**

\[ B = C \circ D \text{ with } D \text{ degree } k \text{ iff there is a B-Poncelet curve } C \text{ such that if } B \text{ identifies } \{z_1, \ldots, z_n\} \in \mathbb{T} \text{ ordered with increasing argument, then } C \text{ is inscribed in the polygon formed joining every } m\text{-th pt.} \]

This needs a new applet! http://lexiteria.com/~ashaffer/blaschke_loci/blaschke.html.
Which one is a composition?

The Poncelet curve associated to a degree-3 Blaschke product is an ellipse:

Figure: Blaschke products of degree 9
Which one is a composition?

The Poncelet “2-curve” associated with a Blaschke product is a pt.

Figure: Blaschke products of degree 8

What you see: Density of indecomposable Blaschke products in the set of finite Blaschke products.
Figure: thanks to G. Semmler and E. Wegert

For more info see: E. Wegert, Visual Complex Functions, 2012
Otto Frostman received his B. Sc. degree from Lund University in Sweden, where he pursued graduate studies under the younger of the two Riesz brothers, Marcel Riesz.


Three Theorems of Frostman: Theorem 1

$I$ inner, analytic on $\mathbb{D}$, radial limits of modulus 1 a.e. on $\mathbb{D}$;

$I = BS$, $B$ (infinite) Blaschke, $S$ inner with no zeros in $\mathbb{D}$.

**Theorem**

Let $I$ be an inner function. Then for all $a \in \mathbb{D}$, except possibly a set of capacity zero, $\varphi_a \circ I$ is a Blaschke product.
Figure: Mystery function
The atomic singular inner function: For better or for worse

Figure: Atomic singular inner function

\[ S(z) = \exp\left(\frac{1+z}{1-z}\right); \varphi_a \circ S \text{ is a Blaschke product for all } a \neq 0. \]

But not at 0, of course.
Doing the Frostman shift

**Theorem (Frostman’s First Theorem)**

Let $I$ be an inner function. Then for all $a \in \mathbb{D}$, except possibly a set of capacity zero, $\varphi_a \circ I$ is a Blaschke product.

Singular inner functions are *rare*:

**Theorem (S. Fisher)**

Let $F$ be a bounded analytic function. The set of $w$ for which $F(z) - w$ has a singular inner factor has logarithmic capacity zero.

When is the Frostman shift of a Blaschke product a Blaschke product?
Indestructible Blaschke products

Some Blaschke products are *indestructible*: $\varphi_a \circ B$ is always a Blaschke products.

Clever name due to Renate McLaughlin (1972) gave necessary and sufficient conditions;
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Examples:

1. Finite Blaschke products;
2. Thin Blaschke products: $\lim_n (1 - |z_n|^2)|B'(z_n)| = 1$;
3. (Kraus & Roth, 2013) Compositions of indestructible Blaschke products; decompositions of indestructible Blaschke products are too.
Non-examples

Recall: $B$ zeros ($z_n$), interpolating if $\inf_n (1 - |z_n|^2) |B'(z_n)| > \delta > 0$
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Alt:

$$\inf_{n: n \neq m} \prod_j \left| \frac{z_n - z_m}{1 - \overline{z_m}z_n} \right| > \delta > 0.$$

$S$, the atomic singular inner fcn, $\varphi_a \circ S$ interpolating for $a \neq 0$.

Note the difference:

$$\inf_n (1 - |z_n|^2)|B'(z_n)| > \delta > 0$$
can be destructible;

$$\lim_n (1 - |z_n|^2)|B'(z_n)| = 1$$
indestructible.
Frostman’s second theorem

Theorem

Let $B$ be an (infinite) Blaschke product with zeros $(a_n)$. Then $B$ and all of $B$’s subproducts have radial limit of modulus one at $\lambda \in \mathbb{T}$ iff

$$\sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{|1 - \overline{a_j} \lambda|} < \infty,$$

Most important set satisfying this condition:

Definition

A Blaschke product is a uniform Frostman Blaschke product if

$$\sup_{\lambda \in \mathbb{T}} \sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{|1 - \overline{a_j} \lambda|} < \infty.$$
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Examples of UFB?

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(Specific Example): $0 < r_n < 1$, $0 < \theta_n < 1$,

$$\sup \left( \frac{\theta_{n+1}}{\theta_n} \right) < 1$$

and

$$\sum_{n=1}^{\infty} \frac{1 - r_n}{\theta_n} < \infty$$

then $(r_n e^{i\theta_n})$ is the zero sequence of a **UFB**.
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Is there a condition depending just on the moduli, like there is for Blaschke products; i.e., $\sum_n (1 - |a_n|)$?
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(Specific Example): \(0 < r_n < 1, \quad 0 < \theta_n < 1,\)

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then \(r_n e^{i\theta_n}\) is the zero sequence of a UFB.

Is there a condition depending just on the moduli, like there is for Blaschke products; i.e., \(\sum_n (1 - |a_n|)\)? No.
If you are allowed to rotate zeros of a Blaschke product, can you always rotate the zeros to obtain a *UFB*?

**Definition**

*A Blaschke product is a uniform Frostman Blaschke product if*

\[
\sup_{\lambda \in \mathbb{T}} \sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{|1 - \frac{a_j}{\lambda}|} < \infty.
\]

(Naftalevitch) You can always rotate to get an interpolating Blaschke product; \( \inf_n (1 - |z_n|^2)|B'(z_n)| \geq \delta > 0. \)
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(Chalendar, Fricain, Timotin) You can always rotate to get a thin Blaschke product; \(\lim_n (1 - |z_n|^2)|B'(z_n)| = 1\).
Theorem (Vasyunin)

\[ B \in UFB \text{ with zeros } (z_n) \implies \sum_n (1 - |z_n|) \log(1/(1 - |z_n|)) < \infty. \]

Theorem (Akeroyd, G)

Let \((r_n)_{n=1}^\infty\) nondecreasing sequence in \([0, 1)\). For there to exist a \(B \in UFB\) with zeros \((z_n)\) having \(|z_n| = r_n\), it is sufficient that there exists \(\varepsilon > 0\) such that the following sum converges:

\[
\sum_{n=1}^\infty (1 - r_n) \log(e/(1 - r_n))[\log(\log(3/(1 - r_n))))]^\varepsilon.
\]
Putting the two theorems together

**Definition**

*A Blaschke product is a UFB if*

\[
\sup_{\lambda \in \mathbb{T}} \sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{|1 - \overline{a_j} \lambda|} < \infty.
\]

Recall: Theorem 1 said \( \varphi_a \circ I \) is almost always a Blaschke product.

**Question 1.** If \( B \) is a uniform Frostman Blaschke product, is \( \varphi_a \circ B \) a Blaschke product?
Putting the two theorems together

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Question 1. If \( B \) is a uniform Frostman Blaschke product, is \( \varphi_a \circ B \) a uniform Frostman Blaschke product?

Question 2. And who cares about UFB anyway?
Definition

µ finite Borel measure (∈ M), the Cauchy transform of µ is

\[(1) \quad (Kµ)(z) = \int_T \frac{1}{1 - \xi z} dµ(ξ), \quad z \in \mathbb{D}\]

\[K = \{Kµ : µ \text{ finite Borel measure}\} \text{ space of Cauchy transforms.}\]

\[\|f\|_K = \inf\{\|µ\| : µ \in M \text{ and } (1) \text{ holds}\} .\]

Definition

φ analytic on \(\mathbb{D}\) is a multiplier if \(f \in K \implies φf \in K\).

Theorem (Hruščev, Vinagradov, 1980)

\(UFB\) is the set of inner functions that are multipliers of \(K\).
Three ways of looking at it:

• If $B \in UFB$, can $B$ be the composition of two infinite Blaschke products? (1994, G, Laroco, Mortini, Rupp)

• When can a composition of multipliers be a multiplier?

• Can a UFB be in the range of a composition operator with a discontinuous inner symbol?
Theorem (Matheson and Ross, CMFT 2007)

If $B \in UFB$, then $\varphi_a \circ B \in UFB$ for all $a \in \mathbb{D}$.

“You can’t Frostman shift your way into (or out of) the class $UFB$”
Theorem (Matheson and Ross, CMFT 2007)

If $B \in UFB$, then $\varphi_a \circ B \in UFB$ for all $a \in \mathbb{D}$.

“You can’t Frostman shift your way into (or out of) the class $UFB$”

Consequence:
We know finite Blaschke products and thin Blaschke products are indestructible. M & R tell us that $UFB$s are too.
Stronger statement: If you post-compose a $UFB$ with $\varphi_a$ you get a $UFB$; what if you postcompose with an infinite Blaschke product?

Example (Akeroyd, G.):

There exists $B \in UFB$ such that $B \circ B \in UFB$.

How do you do it?

Fact: If you're an inner function close (uniformly) to a $UFB$, you're a $UFB$.

Create $B$ so that on a "hot spot" $B \circ B = \prod_{j} B - a_j 1 - a_j B \sim \lambda k B - a_k 1 - a_k B$, a Frostman shift of a $UFB$.
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B \circ B = \prod_j \frac{B - a_j}{1 - \overline{a_j} B} \sim \lambda_k \frac{B - a_k}{1 - \overline{a_k} B},
\]

a Frostman shift of a \( UFB \).
Definition

We say $B$ has angular derivative at $\lambda \in \mathbb{T}$ if for some $\eta \in \mathbb{T}$ the nontangential limit $\angle \lim_{z \to \lambda} \frac{B(z) - \eta}{z - \lambda}$ exists and is finite.

Theorem

A Blaschke product $B$ has angular derivative at a point $\lambda \in \mathbb{T}$ iff

$$\sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{|1 - \overline{a_j}\lambda|^2} < \infty.$$ 

Fact: If you’re an inner function uniformly close to a $BP$ with finite angular derivative at $\lambda$, you have finite angular derivative at $\lambda$ too.
Coincidences?

Thin products: Indestructible,
Thin products: Indestructible, close to thin $\implies$ thin,
Thin products: Indestructible, close to thin $\implies$ thin, finite products of interpolating Blaschke products,
Thin products: Indestructible, close to thin $\implies$ thin, finite
products of interpolating Blaschke products, “close to finite”
Coincidences?

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\textit{UFB}: Indestructible,
Thin products: Indestructible, close to thin $\Rightarrow$ thin, finite products of interpolating Blaschke products, “close to finite”

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$UFB$: Indestructible, close to $UFB \Rightarrow UFB$, finite product of interpolating, feel close to finite and...
Douglas asked: What is the form of a closed subalgebra of $L^\infty$ containing $H^\infty$.

Chang/ Marshall theorem: Every closed subalgebra is of the form $H^\infty[b_\alpha : \alpha \in I, b_\alpha \text{ Blaschke product}]$. 
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**Theorem (Hedenmalm)**

*Let $C$ be a Blaschke product invertible in $H^\infty[B]$ where $B$ is thin. Then $C$ is a finite product of thin Blaschke products.*
Finite angular derivative:

**Theorem**

A Blaschke product $B$ has angular derivative at a point $\lambda \in \mathbb{T}$ iff

$$
\sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{|1 - \overline{a_j}\lambda|^2} < \infty.
$$

**Theorem (Gallardo-Gutierrez, G.)**

Let $C$ be a Blaschke product invertible in $H^\infty[\overline{B}]$ where $B$ has finite angular derivative at $\lambda \in \mathbb{D}$. Then $C$ does too.
Question: If $C$ is a Blaschke product invertible in $H^\infty[\overline{B}]$ where $B \in UFB$, is $C \in UFB$?
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This was “the case in favor.”
Question: If $C$ is a Blaschke product invertible in $H^\infty[\overline{B}]$ where $B$ is a uniform Frostman Blaschke product, is $C \in UFB$?

Suppose $B_1$ is a subproduct of $B$. Then $B = B_1B_2$, so $B_1\overline{B} = B_1(\overline{B_1B_2}) = \overline{B_2}$. Every subproduct is invertible in $H^\infty[\overline{B}]$. But there is a Blaschke product with radial limit of modulus one at a point and some subproduct does not have radial limit at that point.
The case against

Question: If $C$ is a Blaschke product invertible in $H^\infty[\overline{B}]$ where $B$ is a uniform Frostman Blaschke product, is $C \in UFB$?

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But there is a Blaschke product with radial limit of modulus one at a point and some subproduct does not have radial limit at that point.