# Decomposing compositions and three theorems of Frostman

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# Joint work with

Finite Blaschke products: Ueli Daepp, Ben Sokolowsky, Andrew Shaffer, Karl Voss (Bucknell University)

Three theorems of Frostman: John Akeroyd (University of Arkansas, Fayetteville)

Inner function: $I : \mathbb{D} \to \mathbb{D}$ analytic with radial limits of modulus 1 a.e.

**Definition**

*An inner function $I$ is* **indecomposable** *or* **prime** *if whenever $I = U \circ V$ with $U$ and $V$ inner, either $U$ or $V$ is a disk automorphism.*

Question: Which inner functions can be prime?

Motivation from composition operators: $C_\Phi : X \to X$ defined by $C_\Phi(f) = f(\Phi)$.

## One reason to care

Range of composition operators:

**Theorem (J. Ball, 1975; K. Stephenson 1979, (revised))**

*Let $X$ be any $H^p$ space, $0 < p \leq \infty$. and let $M$ be a linear submanifold of $X$ that is closed under uniform convergence on compact subsets of $\mathbb{D}$. Then $M = C_\Phi(X)$ for some inner function $\Phi$, if and only if $M$ has the following properties:*

1. *$M$ contains a nonconstant function.*
2. *If $f, g \in M$ and $f \cdot g \in X$ (resp. $f/g \in X$), then $f \cdot g \in M$ (resp. $f/g \in M$).*
3. *If $f \in M$ and $I$ is the inner factor of $f$, then $I \in M$.*
4. *$M$ contains g.c.d. $\{B \in M : B \text{ inner } B(0) = 0\}$.*

# First: **Finite Blaschke products**

$$B(z) = \lambda \prod_{j=1}^{n} \frac{a_j - z}{1 - \overline{a_j} z}, \text{ where } |a_j| < 1, |\lambda| = 1; \varphi_a(z) = \frac{a - z}{1 - \overline{a} z}.$$

1922-3, J. Ritt reduced to result about groups (Trans. AMS): $F$ is a composition iff the group of $F^{-1}(w)$ is imprimitive.

1974: Carl Cowen gave result for rational functions. (ArXiv)

The group: Associated with the set of covering transformations of the Riemann surface of the inverse of the Blaschke product; Compositions correspond to (proper) normal subgroups.

2000, JLMS Beardon, Ng simplified Ritt's work,
2011 Tsang and Ng, Extended to finite mappings between Riemann surfaces

# Basic Assumptions

$B$ has distinct zeros.

$\varphi_a(z) = (a - z)/(1 - \bar{a}z)$

$B$ is indecomposable iff $\varphi_{B(0)} \circ B$ is, so we suppose $B(0) = 0$.

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$B = C \circ D$ with $C, D$ Blaschke iff $B = (C \circ \varphi_{D(0)}) \circ (\varphi_{D(0)} \circ D)$ is.
So we suppose $B(0) = C(0) = D(0) = 0$.

Nice consequence: $C(z) = zC_1(z)$;
$B(z) = C(D(z)) = D(z)(C_1(D(z)))$ and $D$ is a subfactor of $B$.

# Algorithm 1: Try all sets of zeros ($B$ degree $n$, $B(0) = 0$)

$$\text{Is } B(z) = C \circ D(z)?$$

$$\text{degree}(D) = k, \text{degree}(C) = m, \text{degree}(B) = mk = n$$

Pick subsets of size $k$ to be the zeros of $D$ (include 0) $D$ is $k-\text{to}-1$ so $D$ partitions the zeros of $B$ into $m$ sets of $k$ points. You're done.

**Theorem (Algorithm 1.)**

*$B = C \circ D$ with $D$ degree $k$ iff there is a subproduct $D$ of $B$ of degree $k$ that identifies the zeros of $B$ in $m$ sets of $k$ points.*

But you don't know anything about your Blaschke product.
Won't work for infinite Blaschke products.

# Algorithm 2: Critical Points ($B$ degree $n$, $B(0) = 0$)

Critical point: $B'(z) = 0$; critical value $w = B(z), B'(z) = 0$.

**Theorem (Heins, 1942; Zakeri, BLMS 1998)**

*Let $z_1, \ldots, z_d \in \mathbb{D}$. There exists a unique Blaschke $B$, degree $d+1$, $B(0) = 0$, $B(1) = 1$, and $B'(z_j) = 0$, all $j$.*

**Corollary (Nehari, 1947; Zakeri)**

*Blaschke pdts. $B_1, B_2$ have the same critical pts. iff $B_1 = \varphi_a \circ B_2$ for some automorphism $\varphi_a$.*

**Remark.** $B$ with distinct zeros has $2n - 2$ critical points,

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**Remark.** $B$ with distinct zeros has $2n-2$ critical points, only $n-1$ are in $\mathbb{D}$: $\{z_1, \ldots, z_{n-1}, 1/\overline{z_1}, \ldots, 1/\overline{z_{n-1}}\}$: $B$ has $\leq n-1$ critical values in $\mathbb{D}$.

# Algorithm 2: Counting critical values

$B = C \circ D \implies B'(z) = C'(D(z))D'(z)$; $D$ has $k-1$ critical points, $D$ partitions the others into $m-1$ sets.

**Theorem**

*$B = C \circ D$ iff there exists a subproduct $D$ of $B$ sharing $k-1$ critical pts. with $B$ that partitions the others into $m-1$ sets.*

$B$ can have at most $(k-1)$

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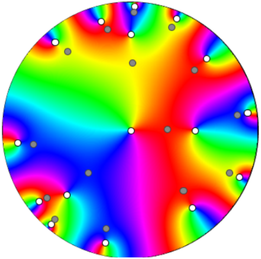
$B$ can have at most $(k-1)+(m-1)$ critical values.

# Which one is a composition?

Note: Argument chooses the color



**Critical Values**

0.01458 +0.00204i (arg=0.14)
0.0229 +0.02531i (arg=0.84)
-0.03201 +0.11138i (arg=1.85) -0.03201 +0.11138i (arg=1.85)
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-0.04863 +0.02413i (arg=2.68) -0.04863 +0.02413i (arg=2.68)
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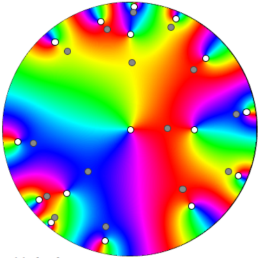

**Critical Values**

0.01465 +0.00205i (arg=0.14) 0.02287 +0.02538i (arg=0.84)
-0.03169 +0.11144i (arg=1.85) -0.03193 +0.11141i (arg=1.85)
-0.03213 +0.11138i (arg=1.85) -0.03228 +0.11132i (arg=1.85)
-0.04849 +0.02415i (arg=2.68) -0.04862 +0.02418i (arg=2.68)
-0.04866 +0.02413i (arg=2.68) -0.04869 +0.02403i (arg=2.68)
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Figure: Blaschke products of degree 16

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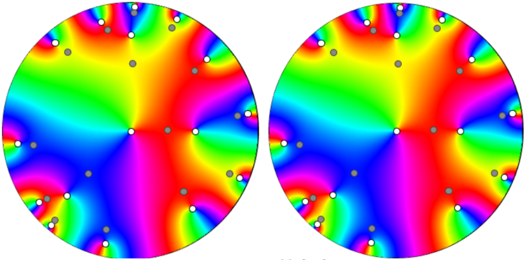

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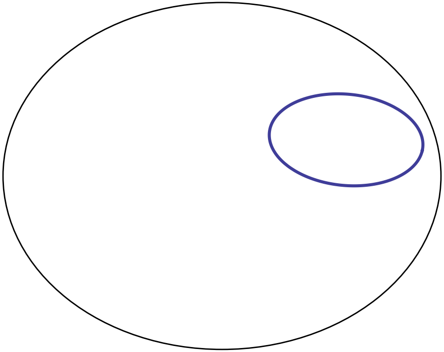
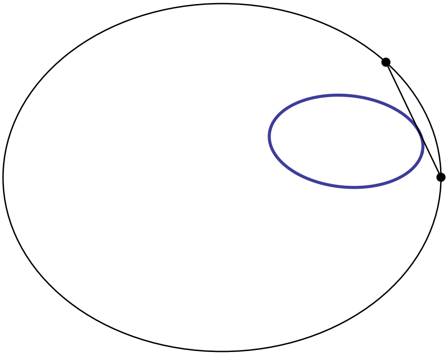
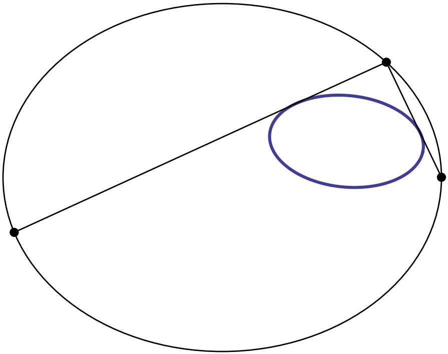
# Algorithm 3: Geometry ($B(0) = 0$, $B$ degree $n = mk$)

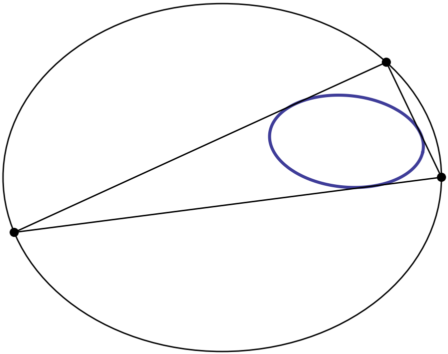
**Theorem (Poncelet's porism)**

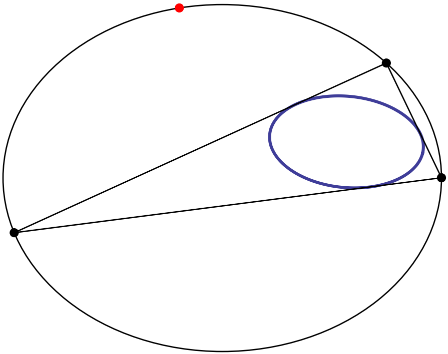
*Let $\mathcal{C}$ and $\mathcal{D}$ be two ellipses. If $\mathcal{C}$ is inscribed in one $n$-gon with vertices on $\mathcal{D}$, then $\mathcal{C}$ is inscribed in every $n$-gon with vertices on $\mathcal{D}$.*

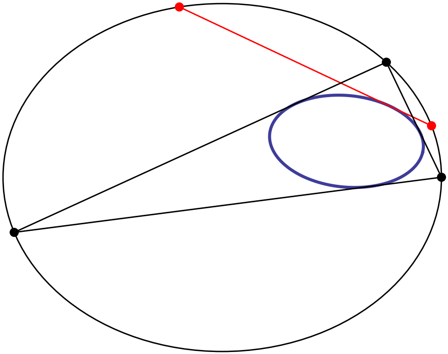


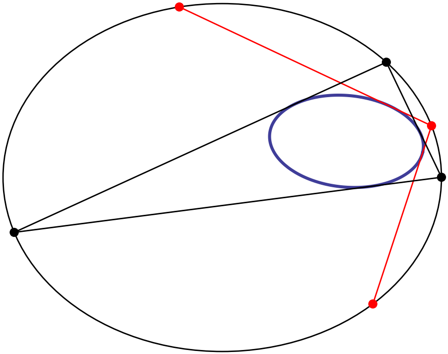


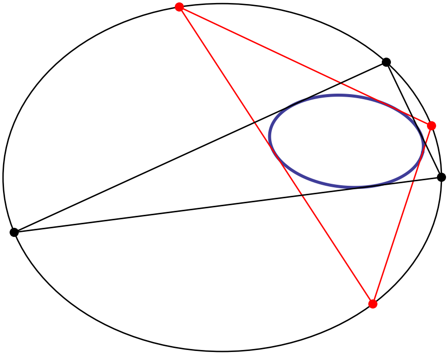












Other things want to be Poncelet curves:

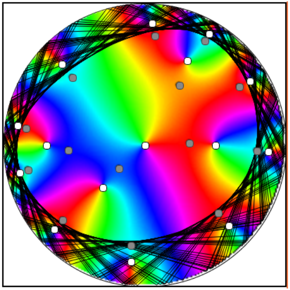


Figure: Acts like a Poncelet curve

## Definition

*$\mathcal{C} \subset \mathbb{D}$ is a* **Poncelet curve** *if whenever $\mathcal{C}$ is inscribed in one $n$-gon with vertices on $\mathbb{T}$, every $\lambda \in \mathbb{T}$ is the vertex of such an $n$-gon.*

Work of Gau-Wu and Daepp, G., Voss implies

**Theorem**

*Every Blaschke product $B$, $B(0) = 0$ degree $n$, is associated with a unique such Poncelet curve; $B$ identifies the vertices of the $n$-gon.*

Applet: Duncan Gillis, Keith Taylor, Thanks to Banach Algebras 2009 `http://www.mscs.dal.ca/~kft/Blaschke/`

# Can we pair Poncelet curves with Blaschke products?

No: Every Blaschke product is associated with a Poncelet curve, but not every Poncelet curve is associated with a Blaschke product. Those that are will be called **B-Poncelet** curves.

**Theorem (DGSSV)**

*$B = C \circ D$ with $D$ degree $k$ iff there is a B-Poncelet curve $\mathcal{C}$ such that if $B$ identifies $\{z_1, \ldots, z_n\} \in \mathbb{T}$ ordered with increasing argument, then $\mathcal{C}$ is inscribed in the polygon formed joining every m-th pt.*

This needs a new applet! `http://lexiteria.com/~ashaffer/blaschke_loci/blaschke.html`.

# Which one is a composition?

The Poncelet curve associated to a degree-3 Blaschke product is an ellipse:

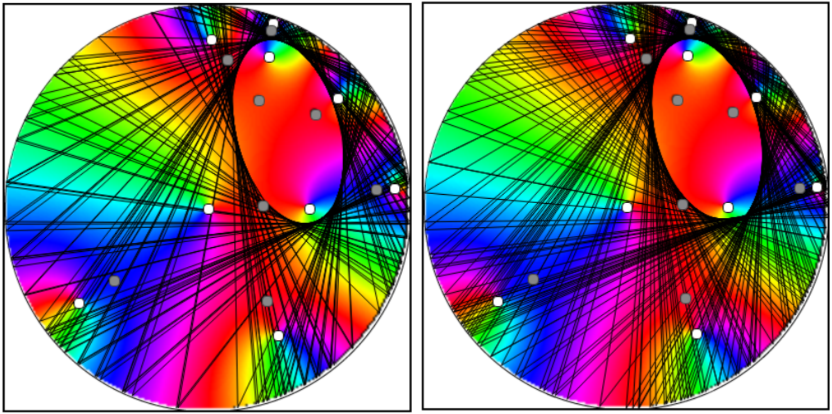


Figure: Blaschke products of degree 9

# Which one is a composition?

The Poncelet "2-curve" associated with a Blaschke product is a pt.

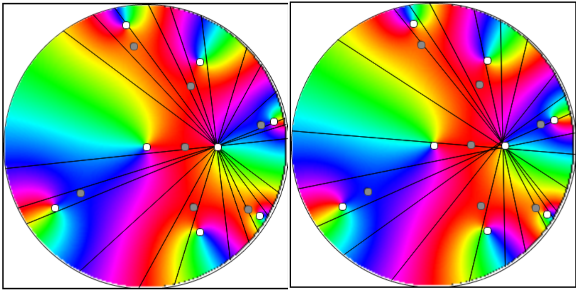


Figure: Blaschke products of degree 8

What you see: Density of indecomposable Blaschke products in the set of finite Blaschke products.

# COMMERCIAL BREAK

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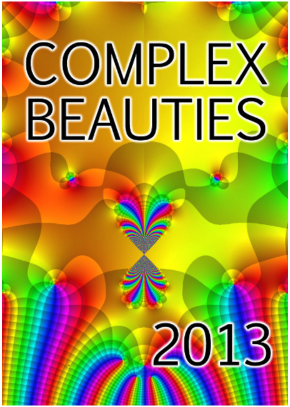


Figure: thanks to G. Semmler and E. Wegert

For more info see: E. Wegert, Visual Complex Functions, 2012

# Otto Frostman

Otto Frostman received his B. Sc. degree from Lund University in Sweden, where he pursued graduate studies under the younger of the two Riesz brothers, Marcel Riesz.

Theorem 1 from Frostman's thesis, *Potential d'équilibre et capacité des ensembles avec guilques applications á la théorie des fonctions*, Medd. Lunds Univ. Mat. Sem. 3, 1935.

Theorems 2 and 3, *Sur les produits de Blaschke*, Knugl. Fysiografiska Sällskapets I Lund Förhandlingar, 1942.

# Three Theorems of Frostman: Theorem 1

$I$ inner, analytic on $\mathbb{D}$, radial limits of modulus 1 a.e. on $\mathbb{D}$;

$I = BS$, $B$ (infinite) Blaschke, $S$ inner with no zeros in $\mathbb{D}$.

**Theorem**

*Let $I$ be an inner function. Then for all $a \in \mathbb{D}$, except possibly a set of capacity zero, $\varphi_a \circ I$ is a Blaschke product.*

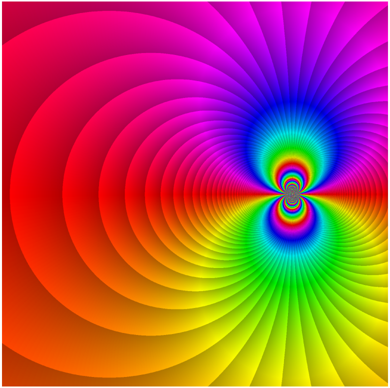

Figure: Mystery function

# The atomic singular inner function: For better or for worse

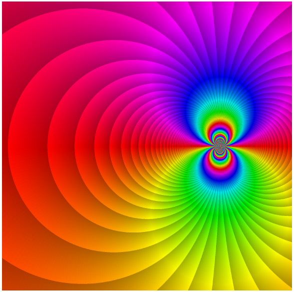

Figure: Atomic singular inner function

$S(z) = \exp\left(\frac{1+z}{1-z}\right)$; $\varphi_a \circ S$ is a Blaschke product for all $a \neq 0$.

But not at 0, of course.

# Doing the Frostman shift

**Theorem (Frostman's First Theorem)**

*Let $I$ be an inner function. Then for all $a \in \mathbb{D}$, except possibly a set of capacity zero, $\varphi_a \circ I$ is a Blaschke product.*

Singular inner functions are *rare*:

**Theorem (S. Fisher)**

*Let $F$ be a bounded analytic function. The set of $w$ for which $F(z) - w$ has a singular inner factor has logarithmic capacity zero.*

When is the Frostman shift of a Blaschke product a Blaschke product?

# Indestructible Blaschke products

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Examples:

1. Finite Blaschke products;
2. Thin Blaschke products: $\lim_n (1 - |z_n|^2)|B'(z_n)| = 1$;
3. (Kraus & Roth, 2013) Compositions of indestructible Blaschke products; decompositions of indestructible Blaschke products are too.

# Non-examples

Recall: $B$ zeros $(z_n)$, interpolating if $\inf_n (1 - |z_n|^2)|B'(z_n)| > \delta > 0$

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Alt:

$$\inf_{n:n\neq m} \prod_j \left| \frac{z_n - z_m}{1 - \overline{z_m} z_n} \right| > \delta > 0.$$

$S$, the atomic singular inner fcn, $\varphi_a \circ S$ interpolating for $a \neq 0$.

Note the difference:

$\inf_n(1-|z_n|^2)|B'(z_n)| > \delta > 0$ can be destructible;
$\lim_n(1-|z_n|^2)|B'(z_n)| = 1$ indestructible.

# Frostman's second theorem

## Theorem

*Let $B$ be an (infinite) Blaschke product with zeros $(a_n)$. Then $B$ and all of $B$'s subproducts have radial limit of modulus one at $\lambda \in \mathbb{T}$ iff*

$$\sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{|1 - \overline{a_j}\lambda|} < \infty,$$

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$$\sum_{j=1}^{\infty} \frac{1-|a_j|^2}{|1-\overline{a_j}\lambda|} < \infty,$$

Most important set satisfying this condition:

**Definition**

*A Blaschke product is a uniform Frostman Blaschke product if*

$$\sup_{\lambda \in \mathbb{T}} \sum_{j=1}^{\infty} \frac{1-|a_j|^2}{|1-\overline{a_j}\lambda|} < \infty.$$

# Examples of UFB?

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(Specific Example): $0 < r_n < 1$, $0 < \theta_n < 1$,

$$\sup \left( \frac{\theta_{n+1}}{\theta_n} \right) < 1$$

and

$$\sum_{n=1}^{\infty} \frac{1 - r_n}{\theta_n} < \infty$$

then $(r_n e^{i\theta_n})$ is the zero sequence of a *UFB*.

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Is there a condition depending just on the moduli, like there is for Blaschke products; i.e., $\sum_n (1 - |a_n|)$? No.

If you are allowed to rotate zeros of a Blaschke product, can you always rotate the zeros to obtain a *UFB*?

**Definition**

*A Blaschke product is a uniform Frostman Blaschke product if*

$$\sup_{\lambda \in \mathbb{T}} \sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{|1 - \overline{a_j}\lambda|} < \infty.$$

(Naftalevitch) You can always rotate to get an interpolating Blaschke product; $\inf_n (1 - |z_n|^2)|B'(z_n)| \geq \delta > 0$.

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(Chalendar, Fricain, Timotin) You can always rotate to get a thin Blaschke product; $\lim_n(1-|z_n|^2)|B'(z_n)|=1$.

**Theorem (Vasyunin)**

$B \in UFB$ *with zeros* $(z_n) \implies \sum_n (1-|z_n|)\log(1/(1-|z_n|)) < \infty$.

**Theorem (Akeroyd, G)**

*Let* $(r_n)_{n=1}^{\infty}$ *nondecreasing sequence in* $[0,1)$. *For there to exist a* $B \in UFB$ *with zeros* $(z_n)$ *having* $|z_n| = r_n$, *it is sufficient that there exists* $\varepsilon > 0$ *such that the following sum converges:*

$$\sum_{n=1}^{\infty} (1-r_n)\log(e/(1-r_n))[\log(\log(3/(1-r_n)))]^{\varepsilon}.$$

# Putting the two theorems together

**Definition**

*A Blaschke product is a UFB if*

$$\sup_{\lambda \in \mathbb{T}} \sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{|1 - \overline{a_j}\lambda|} < \infty.$$

Recall: Theorem 1 said $\varphi_a \circ I$ is almost always a Blaschke product.

Question 1. If $B$ is a uniform Frostman Blaschke product, is $\varphi_a \circ B$ a Blaschke product?

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Question 2. And who cares about UFB anyway?

## We do!

**Definition**

*$\mu$ finite Borel measure ($\in M$), the Cauchy transform of $\mu$ is*

$$(1) \qquad (K\mu)(z) = \int_{\mathbb{T}} \frac{1}{1-\bar{\xi}z}\, d\mu(\xi),\ z \in \mathbb{D}$$

*$\mathcal{K} = \{K\mu : \mu$ finite Borel measure$\}$ space of Cauchy transforms.*

$\|f\|_K = \inf\{\|\mu\| : \mu \in M$ and (1) holds$\}$.

**Definition**

*$\phi$ analytic on $\mathbb{D}$ is a multiplier if $f \in \mathcal{K} \implies \phi f \in \mathcal{K}$.*

**Theorem (Hruščev, Vinagradov, 1980)**

*UFB is the set of inner functions that are multipliers of $\mathcal{K}$.*

# Three ways of looking at it:

• If $B \in UFB$, can $B$ be the composition of two infinite Blaschke products? (1994, G, Laroco, Mortini, Rupp)

• When can a composition of multipliers be a multiplier?

• Can a UFB be in the range of a composition operator with a discontinuous inner symbol?

**Theorem (Matheson and Ross, CMFT 2007)**

*If $B \in UFB$, then $\varphi_a \circ B \in UFB$ for all $a \in \mathbb{D}$.*

"You can't Frostman shift your way into (or out of) the class *UFB*"

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"You can't Frostman shift your way into (or out of) the class *UFB*"

Consequence:
We know finite Blaschke products and thin Blaschke products are indestructible. M & R tell us that *UFB*s are too.

Stronger statement: If you post-compose a *UFB* with $\varphi_a$ you get a *UFB*; what if you postcompose with an infinite Blaschke product?

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**Example (Akeroyd, G.)**

*There exists $B \in UFB$ such that $B \circ B \in UFB$.*

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Fact: If you're an inner function close (uniformly) to a UFB, you're a UFB.

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Fact: If you're an inner function close (uniformly) to a UFB, you're a UFB. Create $B$ so that on a "hot spot"

$$B \circ B = \prod_j \frac{B - a_j}{1 - \overline{a_j} B} \sim \lambda_k \frac{B - a_k}{1 - \overline{a_k} B},$$

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a Frostman shift of a *UFB*.

# One more theorem of Frostman

**Definition**

*We say $B$ has angular derivative at $\lambda \in \mathbb{T}$ if for some $\eta \in \mathbb{T}$ the nontangential limit $\angle \lim_{z \to \lambda} \frac{B(z)-\eta}{z-\lambda}$ exists and is finite.*

**Theorem**

*A Blaschke product $B$ has angular derivative at a point $\lambda \in \mathbb{T}$ iff*

$$\sum_{j=1}^{\infty} \frac{1-|a_j|^2}{|1-\overline{a_j}\lambda|^2} < \infty.$$

Fact: If you're an inner function uniformly close to a $BP$ with finite angular derivative at $\lambda$, you have finite angular derivative at $\lambda$ too.

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*UFB*: Indestructible, close to *UFB* $\implies$ *UFB*, finite product of interpolating, feel close to finite and...

# A Banach algebra point of view

Douglas asked: What is the form of a closed subalgebra of $L^\infty$ containing $H^\infty$.

Chang/Marshall theorem: Every closed subalgebra is of the form $H^\infty[\overline{b_\alpha} : \alpha \in I,\ b_\alpha$ Blaschke product$]$.

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**Theorem (Hedenmalm)**

*Let $C$ be a Blaschke product invertible in $H^\infty[\overline{B}]$ where $B$ is thin. Then $C$ is a finite product of thin Blaschke products.*

Finite angular derivative:

**Theorem**

*A Blaschke product $B$ has angular derivative at a point $\lambda \in \mathbb{T}$ iff*

$$\sum_{j=1}^{\infty} \frac{1 - |a_j|^2}{|1 - \overline{a_j}\lambda|^2} < \infty.$$

**Theorem (Gallardo-Gutierrez, G.)**

*Let $C$ be a Blaschke product invertible in $H^\infty[\overline{B}]$ where $B$ has finite angular derivative at $\lambda \in \mathbb{D}$. Then $C$ does too.*

**Question: If $C$ is a Blaschke product invertible in $H^\infty[\overline{B}]$ where $B \in UFB$, is $C \in UFB$?**

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This was "the case in favor."

# The case against

**Question: If $C$ is a Blaschke product invertible in $H^\infty[\overline{B}]$ where $B$ is a uniform Frostman Blaschke product, is $C \in UFB$?**

Suppose $B_1$ is a subproduct of $B$. Then $B = B_1 B_2$, so $B_1 \overline{B} = B_1(\overline{B_1 B_2}) = \overline{B_2}$. Every subproduct is invertible in $H^\infty[\overline{B}]$.

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But there is a Blaschke product with radial limit of modulus one at a point and some subproduct does *not* have radial limit at that point.