Isomorphisms Invariants for Multivariable C^*-dynamics

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(joint work with E. Katsoulis, to appear in J. Noncom. Geom.)

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2. For multivariable systems, and piecewise conjugacy:
   Davidson-Katsoulis (2007).

Additional Motivation

- Cornelissen and Marcolli: number theory, graph theory, C^*-algebras

[1] G. Cornelissen, *Curves, Dynamical Systems and Weighted Point Counting*, Proc. Nat. Acad. Sc., to appear.

[2] G. Cornelissen and M. Marcolli, *Quantum Statistical Mechanics, L-series and Anabelian Geometry*, preprint arXiv: 1009.0736.

[3] G. Cornelissen and M. Marcolli, *Graph Reconstruction and Quantum Statistical Mechanics*, J. Geom. Phys., to appear.

- Link: Piecewise conjugacy of classical systems is an invariant for tensor algebras, as proved by Davidson-Katsoulis (2007).

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Let's denote such an algebra by $\mathrm{alg}(A, \alpha)$.

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- And $\mathscr{O}_2$ is not $*$-isomorphic to $\mathrm{C}^*(\mathbb{F}_2)$.

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(X, σ) and (Y, τ) are *piecewise conjugate* if $n = n_\alpha = n_\beta$ and there exists a homeomorphism $\phi \colon X \to Y$ and an open cover $\{U_g \mid g \in S_n\}$ of Y s.t.

$$\tau_i|_{U_g} = \phi \circ \sigma_{g(i)} \circ \phi^{-1}|_{U_g}.$$

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- It is not assumed a priori that they have the same multiplicity.

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- The general case is an open problem.

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(ii) $(A,\alpha) \sim (B,\beta)$ (i.e., *conjugate*) if U above is a diagonal up to a permutation $(n_\alpha = n_\beta)$.

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- Converse? Difficult even for $n_\alpha = n_\beta = 1$.

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3. piecewise conjugacy $\Leftarrow$ isomorphic tensor algebras for classical systems

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Summary

1. unitary equivalence $\Rightarrow$ isomorphic tensor algebras
   $\overset{?}{\Leftarrow}$
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3. piecewise conjugacy $\Leftarrow$ isomorphic tensor algebras for classical systems
   $\overset{?}{\Rightarrow}$ Yes, in some cases.

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2. Identify a complete invariant for isomorphisms between operator algebras associated with C*-dynamical systems (At least for classical systems and isometric isomorphisms.)

Question 2

A Complete Invariant.

Stably finite C^*-algebras

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  $\gamma|_A\colon A \to B$ is a $*$-isomorphism (fixes the coefficients).

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- Substitute $[b_{ij}]$ with the unitary of its polar decomposition. $\square$

C^*-algebras with trivial centre by automorphisms

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Let (A,α) and (B,β) be $*$-epimorphic systems. Then:

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- Apply Gaussian Elimination to $[\rho(b_{ij})]$. The inverse must be $[\rho(d_{ij})]$.
- Use the full atomic representation of B to reconstruct the inverse which must be $[d_{ij}]$. □

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Hence $\mathscr{T}_+(\mathscr{O}_2, \alpha) \simeq \mathscr{T}_+(\mathscr{O}_2, \beta)$, yet they have different multiplicities.

Question 1

Piecewise conjugacy for non-classical systems.

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- $\widehat{A}$: the space of un.eq. classes of irreducible rpns of A.
- If $\alpha \in \mathrm{Aut}(A)$ then it induces a homeomorphism $\widehat{\alpha} \colon \widehat{A} \to \widehat{A}$.

Theorem (K.-Katsoulis 2012)

Let (A, α) and (B, β) be automorphic. If $\mathscr{T}_+(A, \alpha) \simeq \mathscr{T}_+(B, \beta)$ then the systems are piecewise conjugate over their Fell spectra.

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  Kadison-Ringrose: (A, α) with $\alpha \in \mathrm{Aut}(A)$ universally weakly inner but not inner.
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Conclusion

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Motivation: For the particular systems in the work of Cornelissen and Marcolli piecewise conjugacy is shown also to be a complete invariant.

Thank You.