Duality results for group von Neumann algebras and jointly invariant operator spaces

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Joint invariance

Consider a (weak-* closed) subspace $\mathcal{U} \subseteq \mathcal{B}(\ell^2(\mathbb{Z}))$ which is

(a) invariant under left and right multiplication by the algebra $\mathcal{D}$ of diagonal matrices

and simultaneously

(b) invariant under 'diagonal translation' by the group $\mathbb{Z}$:
if $[a_{i,j}] \in \mathcal{U}$ *and* $k \in \mathbb{Z}$, *then* $[a_{i+k,j+k}] \in \mathcal{U}$.

Now (a) forces $\mathcal{U}$ to consist of *all* matrices *supported* on some *pattern* $\Omega \subset \mathbb{Z} \times \mathbb{Z}$,

and then (b) forces Ω to be of the form $\{(i,j) : j - i \in E\}$ for an appropriate set $E \subseteq \mathbb{Z}$.

What can we say if $\mathbb{Z}$ is replaced by a more general locally compact, perhaps non-abelian, group G?
First of all, what should we mean by *the support* of a set of operators?

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Supports of operators and masa bimodules

Say $T : L^2(X,\mu) \to L^2(Y,\nu)$ vanishes in a Borel rectangle $A \times B$ whenever $P(B)TP(A) = 0$.

Then T is supported in every set $\Omega \subseteq X \times Y$ 'almost disjoint' from $A \times B$.

This means $\Omega \cap (A \times B) \subseteq M \times Y \cup X \times N$ where $\mu(M) = 0 = \nu(N)$. The set $M \times Y \cup X \times N$ is marginally null.

- Fix $\Omega \subseteq X \times Y$. The set $\mathcal{M} = \mathfrak{M}_{\mathrm{max}}(\Omega)$ of all T which are supported in Ω is

(a) w*-closed

(b) a masa bimodule: $\mathcal{D}_x \mathcal{M} \mathcal{D}_y \subseteq \mathcal{M}$

(c) reflexive, i.e. $\mathcal{M} = \mathcal{R}^\perp$ for a set $\mathcal{R}$ of *rank one operators*.

(cf *Loginov-Shulman*).

Note that a selfadjoint unital algebra is reflexive iff it is a von Neumann algebra.

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- Dually, fix a w*-closed masa bimodule $\mathcal{M}$; 'its support' ought to be the complement of the union of all Borel rectangles on which every $T \in \mathcal{M}$ vanishes. Measurability?

There is a countable family $\mathcal{E}$ of Borel rectangles whose union ω-contains every Borel $A \times B$ s.t. $P(B)\mathcal{M}P(A) = \{0\}$.
The complement of this union (such a set is called ω-closed) is called the ω-support $\operatorname{supp} \mathcal{M}$ of $\mathcal{M}$.

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So, every masa bimodule $\mathcal{M}$ has a well-defined support and $\mathcal{M} \subseteq \mathfrak{M}_{\mathrm{max}}(\operatorname{supp} \mathcal{M})$.

Every *reflexive* masa bimodule is of the form $\mathfrak{M}_{\mathrm{max}}(\Omega)$; and if Ω is ω-closed, then it is unique (modulo marginally null sets) (*Erdos - K - Shulman, 1998*).

NB. The support of any set $\mathcal{S}$ of operators is the same as the support of the smallest w*-closed masa bimodule containing it.

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Supports of operators and masa bimodules

Arveson (1974) defines his supp_A to be closed; so, two different reflexive masa bimodules can have the same closed support.

Example: [1] (for $H = L^2([0,1])$), let $\mathcal{M} := \mathcal{D} + P\mathcal{B}(H)P$ where $P = P(A)$ with $A \subseteq [0,1]$ and A^c meeting every open set nontrivially.
This has the same supp_A with $\mathcal{B}(H)$.

Starting with a reflexive masa bimodule, he *defines* the topology using it.
We just fix the masas, and represent all masa bimodules simultaneously.

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Predual formulation

View $\mathcal{B}(L^2(X), L^2(Y))$ as the dual of the projective tensor product $T(X, Y) := L^2(X)\hat{\otimes}L^2(Y)$.
This can be identified with the space of all functions of the form

$$h(x, y) = \sum_i f_i(x)g_i(y)$$

where $f_i, g_i \in L^2$ and $\sum_i \|f_i\|_2 \|g_i\|_2 < \infty$.
We identify functions which agree marginally almost everywhere, i.e. on a set $M^c \times N^c$ with N, M null sets. [2]

[2] (draw)

Schur multipliers

Henceforth let G be a locally compact 2nd countable group.

A Schur multiplier $w \in \mathfrak{S}(G)$ is a function $w : G \times G \to G$ so that $\quad h \in T(G) \Rightarrow wh \in T(G)$.
Such w defines a bounded operator $m_w : h \to wh$ on $T(G)$.

The map m_w is an $L^\infty(G)$-bimodule map.

Its dual is written $S_w : \mathcal{B}(L^2(G)) \to \mathcal{B}(L^2(G))$.

It can be shown that the set $\{S_w : w \in \mathfrak{S}(G)\}$ is pointwise - weak* generated by *elementary* maps
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Jointly invariant subspaces

Recall the adjoint right action of G on $\mathcal{B}(L^2(G))$ given by $\mathrm{Ad}\rho_r(T) = \rho_r T \rho_r^*$, $(r \in G)$ (so if $T = T_k$ is Hilbert-Schmidt then $\mathrm{Ad}\rho_r(T_k)$ has kernel $k_r(s,t) = k(sr, tr)$).

This integrates to a representation of the measure algebra $M(G)$ as operators on $\mathcal{B}(L^2(G))$:

For $\mu \in M(G)$, define $\Gamma(\mu) : \mathcal{B}(L^2(G)) \to \mathcal{B}(L^2(G))$ by

$$\Gamma(\mu)(T) = \int_G \rho_r T \rho_r^* d\mu(r), \quad T \in \mathcal{B}(L^2(G))$$

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The fixed point set of $\Gamma(M(G))$ is $\mathrm{VN}(G) = \text{w-*span}\{\lambda_s : s \in G\}$. (here $(\lambda_s f)(t) = f(s^{-1}t)$, $f \in L^2(G)$)

The fixed point set of $\{S_w : w \in \mathfrak{S}(G)\}$ is the multiplication masa $\mathcal{D} = \{M_f : f \in L^\infty(G)\}$.

These are 'orthogonal' von Neumann algebras in $\mathcal{B}(L^2(G))$, i.e. $\mathrm{VN}(G) \cap \mathcal{D} = \mathbb{C}I$ and $(\mathrm{VN}(G) \cup \mathcal{D})'' = \mathcal{B}(L^2(G))$.

We characterize subspaces of $\mathcal{B}(L^2(G))$ which are invariant under *both actions* of $\mathfrak{S}(G)$ and $M(G)$:

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Let $\mathcal{U} \subseteq \mathcal{B}(L^2(G))$ *be a weak* closed subspace. The following are equivalent:*
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The Fourier algebra $A(G)$

The Fourier algebra $A(G)$ (*Eymard, 1964*) is the set of all functions $u : G \to \mathbb{C}$ of the form $u(s) = (\lambda_s f, g)$ with $f, g \in L^2(G)$ (here $(\lambda_s f)(t) = f(s^{-1}t)$).

($A(G) \simeq L^1(\hat{G})$ when G is abelian.)

This is a linear space, in fact an algebra of functions on G.

It can be identified with the predual of the von Neumann algebra of G: $\mathrm{VN}(G) = $ w-* span$\{\lambda_s : s \in G\}$.

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Idea of proof: Show that the ideal that does the job is

$$J = \{u \in A(G) : N(u)\chi_{L\times L} \in \mathcal{U}_\perp \text{ for every compact set } L \subseteq G\}$$

where $N(u)(s,t) = u(ts^{-1})$.

Key result: If $J \subseteq A(G)$ closed ideal, $\mathrm{Bim}(J^\perp) = (\mathrm{Sat}(J))^\perp$.

Question: If a weak* closed masa bimodule has support of the form E^* (for some closed $E \subseteq G$) is it necessarily also invariant under all $\mathrm{Ad}\rho_r$?
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Recall: Every *reflexive* masa bimodule is of the form $\mathfrak{M}_{\max}(\Omega)$; and if Ω is ω-closed, then it is unique (modulo marginally null sets) (*Erdos - K - Shulman, 1998*).

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Given an ω-closed set Ω, there is a weak*-closed masa bimodule $\mathfrak{M}_{\min}(\Omega)$ such that

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Synthesis in $A(G)$

For $E \subseteq G$ closed, define

$$I(E) = \{g \in A(G) : g|_E = 0\}$$

and its subset

$$J(E) = \overline{\{g \in A(G) : \operatorname{supp} g \cap E = \emptyset\}}^{\|\cdot\|_A}.$$

Then E is called a set of spectral synthesis if $J(E) = I(E)$.

If G is discrete, then $J(E) = I(E)$ for all E. [3]
If not, there always exists $E \subseteq G$ s.t. $J(E) \subsetneq I(E)$ (*Malliavin, 1959*).

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E. Kaniuth-A.T. Lau (2001): Given a weak* closed invariant subspace $\mathcal{X} \subseteq \mathrm{VN}(G)$, call $E \subseteq G$ $\mathcal{X}$-spectral if $\mathcal{X} \cap I(E) = \mathcal{X} \cap J(E)$.

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Theorem

If $A(G)$ has an approximate identity (bdd or not), a closed set $E \subseteq G$ is $\mathcal{X}$-spectral iff E^* is $(\mathrm{Bim}\,\mathcal{X})$-operator synthetic.

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