Sets of multiplicity in locally compact groups and their operator versions

Ivan Todorov
(joint work with V.S. Shulman and L. Turowska)

31 July 2013
Gothenburg

Content

- Sets of multiplicity in Harmonic Analysis

Content

- Sets of multiplicity in Harmonic Analysis
- Operator versions – why and how

Content

- Sets of multiplicity in Harmonic Analysis
- Operator versions – why and how
- Preservation properties

Content

- Sets of multiplicity in Harmonic Analysis
- Operator versions – why and how
- Preservation properties
- Sets of (operator) p-multiplicity

Sets of multiplicity in $\mathbb{T}$

The Fourier algebra of $\mathbb{T}$:

$$A(\mathbb{T}) = \{\sum_{n\in\mathbb{Z}} c_n e^{int} : \sum_{n\in\mathbb{Z}} |c_n| < \infty\} \simeq \ell^1(\mathbb{Z}).$$

Sets of multiplicity in $\mathbb{T}$

The Fourier algebra of $\mathbb{T}$:

$A(\mathbb{T}) = \{\sum_{n\in\mathbb{Z}} c_n e^{int} : \sum_{n\in\mathbb{Z}} |c_n| < \infty\} \simeq \ell^1(\mathbb{Z})$.

The space of *pseudo-measures* $PM(\mathbb{T}) = A(\mathbb{T})^*$ can be identified with $\ell^\infty(\mathbb{Z})$ via Fourier transform $F \mapsto (\hat{F}(n))_{n\in\mathbb{Z}} \ldots$

Sets of multiplicity in $\mathbb{T}$

The Fourier algebra of $\mathbb{T}$:

$A(\mathbb{T}) = \{\sum_{n\in\mathbb{Z}} c_n e^{int} : \sum_{n\in\mathbb{Z}} |c_n| < \infty\} \simeq \ell^1(\mathbb{Z})$.

The space of *pseudo-measures* $PM(\mathbb{T}) = A(\mathbb{T})^*$ can be identified with $\ell^\infty(\mathbb{Z})$ via Fourier transform $F \mapsto (\hat{F}(n))_{n\in\mathbb{Z}} \ldots$

…and the space of *pseudo-functions* $PF(\mathbb{T}) = \{F \in PM(\mathbb{T}) : \hat{F}(n) \to 0, \text{ as } n \to \infty\}$ is $*$-isomorphic to $c_0(\mathbb{Z})$.

Sets of multiplicity in $\mathbb{T}$

The Fourier algebra of $\mathbb{T}$:

$A(\mathbb{T}) = \{\sum_{n\in\mathbb{Z}} c_n e^{int} : \sum_{n\in\mathbb{Z}} |c_n| < \infty\} \simeq \ell^1(\mathbb{Z})$.

The space of *pseudo-measures* $PM(\mathbb{T}) = A(\mathbb{T})^*$ can be identified with $\ell^\infty(\mathbb{Z})$ via Fourier transform $F \mapsto (\hat{F}(n))_{n\in\mathbb{Z}} \dots$

…and the space of *pseudo-functions* $PF(\mathbb{T}) = \{F \in PM(\mathbb{T}) : \hat{F}(n) \to 0, \text{ as } n \to \infty\}$ is $*$-isomorphic to $c_0(\mathbb{Z})$.

If E is a closed subset of $\mathbb{T}$, let $PM(E)$ denote the space of all pseudo-measures supported on E, $M(E)$ the space of measures $\mu \in M(G)$ with $\operatorname{supp}\mu \subseteq E$, and $N(E)$ the weak* closure of $M(E)$.

Sets of multiplicity in $\mathbb{T}$

The Fourier algebra of $\mathbb{T}$:

$A(\mathbb{T}) = \{\sum_{n\in\mathbb{Z}} c_n e^{int} : \sum_{n\in\mathbb{Z}} |c_n| < \infty\} \simeq \ell^1(\mathbb{Z})$.

The space of *pseudo-measures* $PM(\mathbb{T}) = A(\mathbb{T})^*$ can be identified with $\ell^\infty(\mathbb{Z})$ via Fourier transform $F \mapsto (\hat{F}(n))_{n\in\mathbb{Z}} \ldots$

. . . and the space of *pseudo-functions* $PF(\mathbb{T}) = \{F \in PM(\mathbb{T}) : \hat{F}(n) \to 0, \text{ as } n \to \infty\}$ is $*$-isomorphic to $c_0(\mathbb{Z})$.

If E is a closed subset of $\mathbb{T}$, let $PM(E)$ denote the space of all pseudo-measures supported on E, $M(E)$ the space of measures $\mu \in M(G)$ with $\operatorname{supp}\mu \subseteq E$, and $N(E)$ the weak* closure of $M(E)$.

A closed set $E \subseteq \mathbb{T}$ is an M-set if $PM(E) \cap PF(\mathbb{T}) \neq \{0\} \ldots$

Sets of multiplicity in $\mathbb{T}$

The Fourier algebra of $\mathbb{T}$:

$A(\mathbb{T}) = \{\sum_{n\in\mathbb{Z}} c_n e^{int} : \sum_{n\in\mathbb{Z}} |c_n| < \infty\} \simeq \ell^1(\mathbb{Z})$.

The space of *pseudo-measures* $PM(\mathbb{T}) = A(\mathbb{T})^*$ can be identified with $\ell^\infty(\mathbb{Z})$ via Fourier transform $F \mapsto (\hat{F}(n))_{n\in\mathbb{Z}} \dots$

…and the space of *pseudo-functions* $PF(\mathbb{T}) = \{F \in PM(\mathbb{T}) : \hat{F}(n) \to 0, \text{ as } n \to \infty\}$ is $*$-isomorphic to $c_0(\mathbb{Z})$.

If E is a closed subset of $\mathbb{T}$, let $PM(E)$ denote the space of all pseudo-measures supported on E, $M(E)$ the space of measures $\mu \in M(G)$ with $\operatorname{supp}\mu \subseteq E$, and $N(E)$ the weak* closure of $M(E)$.

A closed set $E \subseteq \mathbb{T}$ is an M-set if $PM(E) \cap PF(\mathbb{T}) \neq \{0\} \dots$

…and an M_1*-set* if $N(E) \cap PF(\mathbb{T}) \neq \{0\}$.

Generalisation to non-commutative groups

G a locally compact second countable group.

$\lambda : L^1(G) \to \mathcal{B}(L^2(G))$ the left regular representation.

$C_r^*(G) = \overline{\lambda(L^1(G))}^{\|\cdot\|_{\mathrm{op}}}$ the reduced C*-algebra of G.

$\mathrm{VN}(G) = \overline{C_r^*(G)}^{w^*}$ the von Neumann algebra of G.

$A(G) = \{s \to (\lambda_s(f), g) : f, g \in L^2(G)\}$ the Fourier algebra of G.

Note $A(G)^* \cong \mathrm{VN}(G)$, $\quad \langle \lambda(f), u \rangle = \int_G f(s)u(s)ds$.

Generalisation to non-commutative groups

G a locally compact second countable group.

$\lambda : L^1(G) \to \mathcal{B}(L^2(G))$ the left regular representation.

$C_r^*(G) = \overline{\lambda(L^1(G))}^{\|\cdot\|_{\mathrm{op}}}$ the reduced C*-algebra of G.

$\mathrm{VN}(G) = \overline{C_r^*(G)}^{w^*}$ the von Neumann algebra of G.

$A(G) = \{s \to (\lambda_s(f), g) : f, g \in L^2(G)\}$ the Fourier algebra of G.

Note $A(G)^* \cong \mathrm{VN}(G)$, $\quad \langle \lambda(f), u \rangle = \int_G f(s)u(s)ds$.

Given a subset $J \subseteq A(G)$, let
$\mathrm{null}(J) = \{s \in G : u(s) = 0, \text{ for all } u \in J\}$ be the null set of J.

Generalisation to non-commutative groups

G a locally compact second countable group.

$\lambda : L^1(G) \to \mathcal{B}(L^2(G))$ the left regular representation.

$C_r^*(G) = \overline{\lambda(L^1(G))}^{\|\cdot\|_{\mathrm{op}}}$ the reduced C*-algebra of G.

$\mathrm{VN}(G) = \overline{C_r^*(G)}^{w^*}$ the von Neumann algebra of G.

$A(G) = \{s \to (\lambda_s(f), g) : f, g \in L^2(G)\}$ the Fourier algebra of G.

Note $A(G)^* \cong \mathrm{VN}(G)$, $\quad \langle \lambda(f), u \rangle = \int_G f(s)u(s)ds$.

Given a subset $J \subseteq A(G)$, let
$\mathrm{null}(J) = \{s \in G : u(s) = 0, \text{ for all } u \in J\}$ be the null set of J.

Given a closed subset $E \subseteq G$, let $I(E)$ (resp. $J(E)$) be the maximal (resp. minimal) closed ideal of $A(G)$ with null set E.

Generalisation to non-commutative groups

Definition

A closed subset $E \subseteq G$ will be called

(i) an *M-set* if $J(E)^{\perp} \cap C_r^*(G) \neq \{0\}$;

(ii) an *M_1-set* if $I(E)^{\perp} \cap C_r^*(G) \neq \{0\}$;

The set E will be called a *U-set* (resp. a *U_1-set*) if it is not an M-set (resp. an M_1-set).

(i) given by M. Bożejko in 1977.

Schur multipliers

Let (X, μ) and (Y, ν) be standard measure spaces.

For a function $\varphi \in L^\infty(X \times Y)$, let $S_\varphi : L^2(X \times Y) \to L^2(X \times Y)$ be the corresponding multiplication operator

$$S_\varphi \xi = \varphi \xi.$$

Schur multipliers

Let (X,μ) and (Y,ν) be standard measure spaces.
For a function $\varphi \in L^\infty(X \times Y)$, let $S_\varphi : L^2(X \times Y) \to L^2(X \times Y)$ be the corresponding multiplication operator

$$S_\varphi \xi = \varphi \xi.$$

The space $L^2(X \times Y)$ can be identified with the Hilbert-Schmidt class in $\mathcal{B}(L^2(X), L^2(Y))$ by

$$\xi \longrightarrow T_\xi, \qquad T_\xi f(y) = \int_X \xi(x,y) f(x) d\mu(x).$$

Set $\|\xi\|_{\mathrm{op}} = \|T_\xi\|_{\mathrm{op}}$

Schur multipliers

Let (X, μ) and (Y, ν) be standard measure spaces.
For a function $\varphi \in L^\infty(X \times Y)$, let $S_\varphi : L^2(X \times Y) \to L^2(X \times Y)$ be the corresponding multiplication operator

$$S_\varphi \xi = \varphi \xi.$$

The space $L^2(X \times Y)$ can be identified with the Hilbert-Schmidt class in $\mathcal{B}(L^2(X), L^2(Y))$ by

$$\xi \longrightarrow T_\xi, \qquad T_\xi f(y) = \int_X \xi(x, y) f(x) d\mu(x).$$

Set $\|\xi\|_{\mathrm{op}} = \|T_\xi\|_{\mathrm{op}}$
A function $\varphi \in L^\infty(X \times Y)$ is called a *Schur multiplier* if there exists $C > 0$ such that

$$\|S_\varphi \xi\|_{\mathrm{op}} \leq C \|\xi\|_{\mathrm{op}}, \quad \xi \in L^2(X \times Y).$$

Closable multipliers

If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, $D \subseteq \mathcal{X}$ a linear subspace (not necessarily closed), and $T : D \to \mathcal{Y}$ is a linear operator then T is called *closable* if for all $y \in \mathcal{Y}$, we have

$$\|x_i\| \to 0, \ \|Tx_i - y\| \to 0 \implies y = 0.$$

Closable multipliers

If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, $D \subseteq \mathcal{X}$ a linear subspace (not necessarily closed), and $T : D \to \mathcal{Y}$ is a linear operator then T is called *closable* if for all $y \in \mathcal{Y}$, we have

$$\|x_i\| \to 0, \ \|Tx_i - y\| \to 0 \implies y = 0.$$

For a measurable function $\varphi : X \times Y \to \mathbb{C}$, let $D(\varphi) = \{\xi \in L^2(X \times Y) : \varphi\xi \in L^2(X \times Y)\}$.

Closable multipliers

If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, $D \subseteq \mathcal{X}$ a linear subspace (not necessarily closed), and $T : D \to \mathcal{Y}$ is a linear operator then T is called *closable* if for all $y \in \mathcal{Y}$, we have

$$\|x_i\| \to 0,\ \|Tx_i - y\| \to 0 \implies y = 0.$$

For a measurable function $\varphi : X \times Y \to \mathbb{C}$, let
$D(\varphi) = \{\xi \in L^2(X \times Y) : \varphi\xi \in L^2(X \times Y)\}$.

Let $S_\varphi : D(\varphi) \to L^2(X \times Y)$ be given by

$$S_\varphi \xi = \varphi\xi.$$

Closable multipliers

If $\mathcal{X}$ and $\mathcal{Y}$ are Banach spaces, $D \subseteq \mathcal{X}$ a linear subspace (not necessarily closed), and $T : D \to \mathcal{Y}$ is a linear operator then T is called *closable* if for all $y \in \mathcal{Y}$, we have

$$\|x_i\| \to 0, \ \|Tx_i - y\| \to 0 \implies y = 0.$$

For a measurable function $\varphi : X \times Y \to \mathbb{C}$, let $D(\varphi) = \{\xi \in L^2(X \times Y) : \varphi\xi \in L^2(X \times Y)\}$.

Let $S_\varphi : D(\varphi) \to L^2(X \times Y)$ be given by

$$S_\varphi \xi = \varphi \xi.$$

Set $\mathfrak{S}_{\rm cl} = \{\varphi : S_\varphi \text{ is closable}\}$.

Call the element of $\mathfrak{S}_{\rm cl}$ *closable multipliers*.

A class of closable multipliers

Theorem (Peller)

φ is a Schur multiplier iff $\exists\ a_k, b_k$ such that
$\left\|\sum_{k=1}^{\infty} |a_k|^2\right\|_\infty < \infty, \quad \left\|\sum_{k=1}^{\infty} |b_k|^2\right\|_\infty < \infty,$

$$\varphi(x,y) = \sum_{k=1}^{\infty} a_k(x) b_k(y), \quad \text{a.e. on } X \times Y.$$

A class of closable multipliers

Theorem (Peller)

φ is a Schur multiplier iff $\exists\ a_k, b_k$ such that
$\left\|\sum_{k=1}^{\infty} |a_k|^2\right\|_\infty < \infty, \quad \left\|\sum_{k=1}^{\infty} |b_k|^2\right\|_\infty < \infty,$

$$\varphi(x,y) = \sum_{k=1}^{\infty} a_k(x) b_k(y), \quad \text{a.e. on } X \times Y.$$

Theorem (Shulman-T-Turowska)

Let $\mathcal{V}$ be the set of all φ that can be represented as above but under the more general conditions

$$\sum_{k=1}^{\infty} |a_k(x)|^2 < \infty, \quad \sum_{k=1}^{\infty} |b_k(y)|^2 < \infty \quad (x \in X, y \in Y).$$

Then $\mathfrak{S}_{\rm cl}$ contains all quotients $\frac{\varphi}{\psi}$, with $\varphi, \psi \in \mathcal{V}$.

Pseudo-topologies and supports

W. Arveson (1974), J. Erdos, A. Katavolos, V.S. Shulman (1998)
Let $\kappa \subseteq X \times Y$.

Pseudo-topologies and supports

W. Arveson (1974), J. Erdos, A. Katavolos, V.S. Shulman (1998)
Let $\kappa \subseteq X \times Y$.

(i) κ is called *marginally null* (denoted $\kappa \simeq \emptyset$) if $\kappa \subseteq (M \times Y) \cup (X \times N)$, where M and N are null.

Pseudo-topologies and supports

W. Arveson (1974), J. Erdos, A. Katavolos, V.S. Shulman (1998)
Let $\kappa \subseteq X \times Y$.

(i) κ is called *marginally null* (denoted $\kappa \simeq \emptyset$) if $\kappa \subseteq (M \times Y) \cup (X \times N)$, where M and N are null.

(ii) κ is said to be *marginally equivalent* to another subset $\kappa' \subseteq X \times Y$ if $\kappa \Delta \kappa'$ is marginally null.

Pseudo-topologies and supports

W. Arveson (1974), J. Erdos, A. Katavolos, V.S. Shulman (1998)
Let $\kappa \subseteq X \times Y$.

(i) κ is called *marginally null* (denoted $\kappa \simeq \emptyset$) if $\kappa \subseteq (M \times Y) \cup (X \times N)$, where M and N are null.

(ii) κ is said to be *marginally equivalent* to another subset $\kappa' \subseteq X \times Y$ if $\kappa \Delta \kappa'$ is marginally null.

(iii) κ is called *ω-open* if κ is marginally equivalent to subset of the form $\cup_{i=1}^{\infty} \kappa_i$, where the sets κ_i are rectangles.

Pseudo-topologies and supports

W. Arveson (1974), J. Erdos, A. Katavolos, V.S. Shulman (1998)
Let $\kappa \subseteq X \times Y$.

(i) κ is called *marginally null* (denoted $\kappa \simeq \emptyset$) if $\kappa \subseteq (M \times Y) \cup (X \times N)$, where M and N are null.

(ii) κ is said to be *marginally equivalent* to another subset $\kappa' \subseteq X \times Y$ if $\kappa \Delta \kappa'$ is marginally null.

(iii) κ is called *ω-open* if κ is marginally equivalent to subset of the form $\cup_{i=1}^{\infty} \kappa_i$, where the sets κ_i are rectangles.

(iv) κ is called *ω-closed* if κ^c is ω-open.

Pseudo-topologies and supports

W. Arveson (1974), J. Erdos, A. Katavolos, V.S. Shulman (1998)
Let $\kappa \subseteq X \times Y$.

(i) κ is called *marginally null* (denoted $\kappa \simeq \emptyset$) if $\kappa \subseteq (M \times Y) \cup (X \times N)$, where M and N are null.

(ii) κ is said to be *marginally equivalent* to another subset $\kappa' \subseteq X \times Y$ if $\kappa \Delta \kappa'$ is marginally null.

(iii) κ is called *ω-open* if κ is marginally equivalent to subset of the form $\cup_{i=1}^{\infty} \kappa_i$, where the sets κ_i are rectangles.

(iv) κ is called *ω-closed* if κ^c is ω-open.

(v) An operator $T \in \mathcal{B}(L^2(X), L^2(Y))$ is *supported on* κ if

$$(\alpha \times \beta) \cap \kappa \simeq \emptyset \;\Rightarrow\; P(\beta)TP(\alpha) = 0,$$

where $P(\alpha)$ is the projection from $L^2(X)$ onto $L^2(\alpha)$.

Masa-bimodules

If $f \in L^\infty(X)$, let $M_f \in \mathcal{B}(L^2(X))$ be the operator of multiplication by f. Set

$$\mathcal{D}_X = \{M_f : f \in L^\infty(X)\};$$

similarly $\mathcal{D}_Y$.

Masa-bimodules

If $f \in L^\infty(X)$, let $M_f \in \mathcal{B}(L^2(X))$ be the operator of multiplication by f. Set

$$\mathcal{D}_X = \{M_f : f \in L^\infty(X)\};$$

similarly $\mathcal{D}_Y$.

$\mathcal{U} \subseteq \mathcal{B}(L^2(X), L^2(Y))$ *masa-bimodule* if $\mathcal{D}_Y \mathcal{U} \mathcal{D}_X \subseteq \mathcal{U}$.

Masa-bimodules

If $f \in L^\infty(X)$, let $M_f \in \mathcal{B}(L^2(X))$ be the operator of multiplication by f. Set

$$\mathcal{D}_X = \{M_f : f \in L^\infty(X)\};$$

similarly $\mathcal{D}_Y$.

$\mathcal{U} \subseteq \mathcal{B}(L^2(X), L^2(Y))$ *masa-bimodule* if $\mathcal{D}_Y \mathcal{U} \mathcal{D}_X \subseteq \mathcal{U}$.

The weak* closed masa-bimodules are precisely the weak* closed invariant spaces of Schur multipliers.

Masa-bimodules

If $f \in L^\infty(X)$, let $M_f \in \mathcal{B}(L^2(X))$ be the operator of multiplication by f. Set

$$\mathcal{D}_X = \{M_f : f \in L^\infty(X)\};$$

similarly $\mathcal{D}_Y$.

$\mathcal{U} \subseteq \mathcal{B}(L^2(X), L^2(Y))$ *masa-bimodule* if $\mathcal{D}_Y \mathcal{U} \mathcal{D}_X \subseteq \mathcal{U}$.

The weak* closed masa-bimodules are precisely the weak* closed invariant spaces of Schur multipliers.

Theorem (Arveson)

Given an ω-closed subset $\kappa \subseteq X \times Y$, there exists a maximal weak* closed masa-bimodule $\mathfrak{M}_{\max}(\kappa)$ and a minimal weak* closed masa-bimodule $\mathfrak{M}_{\min}(\kappa)$ with support κ.

Conditions related to closability of multipliers

Given $\varphi : X \times Y \to \mathbb{C}$, let

$$D^*(\varphi) = \{h \in L^2(X)\hat{\otimes}L^2(Y) : \varphi h \in h \in L^2(X)\hat{\otimes}L^2(Y)\}.$$

Conditions related to closability of multipliers

Given $\varphi : X \times Y \to \mathbb{C}$, let

$$D^*(\varphi) = \{h \in L^2(X)\hat{\otimes}L^2(Y) : \varphi h \in h \in L^2(X)\hat{\otimes}L^2(Y)\}.$$

Let $\kappa_\varphi \subseteq X \times Y$ be the zero set of $D^*(\varphi)$:

$$\kappa_\varphi = \{(x, y) : h(x, y) = 0, \text{ for all } h \in D^*(\varphi)\}.$$

Conditions related to closability of multipliers

Given $\varphi : X \times Y \to \mathbb{C}$, let

$$D^*(\varphi) = \{h \in L^2(X)\hat{\otimes}L^2(Y) : \varphi h \in h \in L^2(X)\hat{\otimes}L^2(Y)\}.$$

Let $\kappa_\varphi \subseteq X \times Y$ be the zero set of $D^*(\varphi)$:

$$\kappa_\varphi = \{(x,y) : h(x,y) = 0, \text{ for all } h \in D^*(\varphi)\}.$$

Theorem

Let $\varphi \in \mathfrak{B}(X \times Y)$.

(i) If $\mathfrak{M}_{\max}(\kappa_\varphi)$ does not contain a compact operator then φ is a closable multiplier;

(ii) If $\mathfrak{M}_{\min}(\kappa_\varphi)$ contains a compact operator then φ is a not closable multiplier.

Sets of operator multiplicity

Let (X, μ) and (Y, ν) be standard measure spaces, $H_1 = L^2(X)$, $H_2 = L^2(Y)$, $\mathcal{K}$ the space of all compact operators from H_1 to H_2.

Definition

An ω-closed set $\kappa \subseteq X \times Y$ is called

(i) an *operator M-set* if $\mathcal{K} \cap \mathfrak{M}_{\max}(\kappa) \neq \{0\}$;

(ii) an *operator M_1-set* if $\mathcal{K} \cap \mathfrak{M}_{\min}(\kappa) \neq \{0\}$.

Sets of operator multiplicity

Let (X, μ) and (Y, ν) be standard measure spaces, $H_1 = L^2(X)$, $H_2 = L^2(Y)$, $\mathcal{K}$ the space of all compact operators from H_1 to H_2.

Definition

An ω-closed set $\kappa \subseteq X \times Y$ is called

(i) an *operator M-set* if $\mathcal{K} \cap \mathfrak{M}_{\max}(\kappa) \neq \{0\}$;

(ii) an *operator M_1-set* if $\mathcal{K} \cap \mathfrak{M}_{\min}(\kappa) \neq \{0\}$.

Studied first by Froelich (1988) in relation with problems in invariant subspace theory.

Sets of operator multiplicity

Let (X,μ) and (Y,ν) be standard measure spaces, $H_1 = L^2(X)$, $H_2 = L^2(Y)$, $\mathcal{K}$ the space of all compact operators from H_1 to H_2.

Definition

An ω-closed set $\kappa \subseteq X \times Y$ is called

(i) an *operator M-set* if $\mathcal{K} \cap \mathfrak{M}_{\max}(\kappa) \neq \{0\}$;

(ii) an *operator M_1-set* if $\mathcal{K} \cap \mathfrak{M}_{\min}(\kappa) \neq \{0\}$.

Studied first by Froelich (1988) in relation with problems in invariant subspace theory.

The space $\mathcal{K}$ is a suitable substitute of $C_r^*(G)$ because of the Stone-von Neumann Theorem:

$$\mathcal{K}(L^2(G)) = \overline{\{M_a T M_b : a \in C_0(G), T \in C_r^*(G)\}}^{\|\cdot\|}.$$

Sets of Toeplitz type

If $E \subseteq G$, let $E^* = \{(s,t) : ts^{-1} \in E\} \subseteq G \times G$.

Sets of Toeplitz type

If $E \subseteq G$, let $E^* = \{(s,t) : ts^{-1} \in E\} \subseteq G \times G$.

Theorem

Let $E \subseteq G$ be a closed set.

(i) E is an M-set if and only if E^* is an operator M-set;

(ii) E is an M_1-set if and only if E^* is an operator M_1-set.

Sets of Toeplitz type

If $E \subseteq G$, let $E^* = \{(s,t) : ts^{-1} \in E\} \subseteq G \times G$.

Theorem

Let $E \subseteq G$ be a closed set.
(i) E is an M-set if and only if E^* is an operator M-set;
(ii) E is an M_1-set if and only if E^* is an operator M_1-set.

Let $T(G) = L^2(G)\hat{\otimes}L^2(G) \equiv (\mathcal{B}(L^2(G))_*$.

If $\varphi \in T(G)$, there exists a (unique) normal completely bounded map $E_\varphi : \mathcal{B}(L^2(G)) \to \mathrm{VN}(G)$ such that

$$\langle E_\varphi(T), u\rangle = \langle T, \varphi N(u)\rangle, \quad u \in A(G).$$

Sets of Toeplitz type

If $E \subseteq G$, let $E^* = \{(s,t) : ts^{-1} \in E\} \subseteq G \times G$.

Theorem

Let $E \subseteq G$ be a closed set.

(i) E is an M-set if and only if E^* is an operator M-set;

(ii) E is an M_1-set if and only if E^* is an operator M_1-set.

Let $T(G) = L^2(G)\hat{\otimes}L^2(G) \equiv (\mathcal{B}(L^2(G))_*$.

If $\varphi \in T(G)$, there exists a (unique) normal completely bounded map $E_\varphi : \mathcal{B}(L^2(G)) \to \mathrm{VN}(G)$ such that

$$\langle E_\varphi(T), u\rangle = \langle T, \varphi N(u)\rangle, \quad u \in A(G).$$

Here $N(u)(s,t) = u(ts^{-1})$.

Sets of Toeplitz type

If $E \subseteq G$, let $E^* = \{(s,t) : ts^{-1} \in E\} \subseteq G \times G$.

Theorem

Let $E \subseteq G$ be a closed set.
(i) E is an M-set if and only if E^* is an operator M-set;
(ii) E is an M_1-set if and only if E^* is an operator M_1-set.

Let $T(G) = L^2(G)\hat{\otimes}L^2(G) \equiv (\mathcal{B}(L^2(G))_*$.

If $\varphi \in T(G)$, there exists a (unique) normal completely bounded map $E_\varphi : \mathcal{B}(L^2(G)) \to \mathrm{VN}(G)$ such that

$$\langle E_\varphi(T), u\rangle = \langle T, \varphi N(u)\rangle, \quad u \in A(G).$$

Here $N(u)(s,t) = u(ts^{-1})$.

If $T \in \mathcal{K}$ then $E_\varphi(T) \in C_r^*(G)$, for every $\varphi \in T(G)$.

Preservation properties – unions

Theorem

If $\kappa_1, \kappa_2 \subseteq X \times Y$ are not operator M-sets (resp. not operator M_1-sets) then $\kappa_1 \cup \kappa_2$ is not an operator M-set (resp. not an operator M_1-set).

Preservation properties – unions

Theorem

If $\kappa_1, \kappa_2 \subseteq X \times Y$ are not operator M-sets (resp. not operator M_1-sets) then $\kappa_1 \cup \kappa_2$ is not an operator M-set (resp. not an operator M_1-set).

Thus, if $E_1, E_2 \subseteq G$ are not M-sets (resp. not M_1-sets) then $\kappa_1 \cup \kappa_2$ is not an M-set (resp. not an M_1-set).

Preservation properties – unions

Theorem

If $\kappa_1, \kappa_2 \subseteq X \times Y$ are not operator M-sets (resp. not operator M_1-sets) then $\kappa_1 \cup \kappa_2$ is not an operator M-set (resp. not an operator M_1-set).

Thus, if $E_1, E_2 \subseteq G$ are not M-sets (resp. not M_1-sets) then $\kappa_1 \cup \kappa_2$ is not an M-set (resp. not an M_1-set).

Question

Does the same holds for countable unions?

Preservation properties – unions

Theorem

If $\kappa_1, \kappa_2 \subseteq X \times Y$ are not operator M-sets (resp. not operator M_1-sets) then $\kappa_1 \cup \kappa_2$ is not an operator M-set (resp. not an operator M_1-set).

Thus, if $E_1, E_2 \subseteq G$ are not M-sets (resp. not M_1-sets) then $\kappa_1 \cup \kappa_2$ is not an M-set (resp. not an M_1-set).

Question

Does the same holds for countable unions?

The proof is based on an intersection result for subspaces of $T(G)$:

Preservation properties – unions

Theorem

If $\kappa_1, \kappa_2 \subseteq X \times Y$ are not operator M-sets (resp. not operator M_1-sets) then $\kappa_1 \cup \kappa_2$ is not an operator M-set (resp. not an operator M_1-set).

Thus, if $E_1, E_2 \subseteq G$ are not M-sets (resp. not M_1-sets) then $\kappa_1 \cup \kappa_2$ is not an M-set (resp. not an M_1-set).

Question

Does the same holds for countable unions?

The proof is based on an intersection result for subspaces of $T(G)$:

If $\mathcal{V}_1, \mathcal{V}_2 \subseteq T(G)$ are dense masa-bimodules then their intersection $\mathcal{V}_1 \cap \mathcal{V}_2$ is also dense.

Preservation properties – products

Theorem

If $\kappa_i \subseteq X_i \times Y_i$ are operator M-sets, $i = 1, 2$, (resp. operator M_1-sets) then $\kappa_1 \times \kappa_2$ is an operator M-set (resp. an operator M_1-set).

Preservation properties – products

Theorem

If $\kappa_i \subseteq X_i \times Y_i$ are operator M-sets, $i = 1, 2$, (resp. operator M_1-sets) then $\kappa_1 \times \kappa_2$ is an operator M-set (resp. an operator M_1-set).

Thus, if $E_i \subseteq G_i$ are M-sets, $i = 1, 2$ (resp. M_1-sets) then $E_1 \cup E_2$ is an M-set (resp. an M_1-set).

Preservation properties – products

Theorem

If $\kappa_i \subseteq X_i \times Y_i$ are operator M-sets, $i = 1, 2$, (resp. operator M_1-sets) then $\kappa_1 \times \kappa_2$ is an operator M-set (resp. an operator M_1-set).

Thus, if $E_i \subseteq G_i$ are M-sets, $i = 1, 2$ (resp. M_1-sets) then $E_1 \cup E_2$ is an M-set (resp. an M_1-set).

The proof of (ii) is based on the following tensor product formula:

$$\mathfrak{M}_{\min}(\kappa_1)\bar{\otimes}\mathfrak{M}_{\min}(\kappa_2) = \mathfrak{M}_{\min}(\kappa_1 \times \kappa_2),$$

which answers a question of Froelich's (1988).

Preservation properties – products

Theorem

If $\kappa_i \subseteq X_i \times Y_i$ are operator M-sets, $i = 1, 2$, (resp. operator M_1-sets) then $\kappa_1 \times \kappa_2$ is an operator M-set (resp. an operator M_1-set).

Thus, if $E_i \subseteq G_i$ are M-sets, $i = 1, 2$ (resp. M_1-sets) then $E_1 \cup E_2$ is an M-set (resp. an M_1-set).

The proof of (ii) is based on the following tensor product formula:

$$\mathfrak{M}_{\min}(\kappa_1)\bar{\otimes}\mathfrak{M}_{\min}(\kappa_2) = \mathfrak{M}_{\min}(\kappa_1 \times \kappa_2),$$

which answers a question of Froelich's (1988).

We use the notion of *pseudo-integral operators*: for measures σ on $Y \times X$ with $|\sigma|_X \leq c\mu$ and $|\sigma|_Y \leq c\nu$,

$$(T_\sigma \xi, \eta) = \int_{Y\times X} \xi(x)\overline{\eta(y)}d\sigma(y, x).$$

Preservation properties – inverse images

Let $\varphi : X \to X_1$, $\psi : Y \to Y_1$ be measurable maps.

Preservation properties – inverse images

Let $\varphi : X \to X_1$, $\psi : Y \to Y_1$ be measurable maps.
For $\kappa_1 \subseteq X_1 \times Y_1$ let

$$\kappa = \{(x, y) : (\varphi(x), \psi(y)) \in X_1 \times Y_1\}.$$

Preservation properties – inverse images

Let $\varphi : X \to X_1$, $\psi : Y \to Y_1$ be measurable maps.
For $\kappa_1 \subseteq X_1 \times Y_1$ let

$$\kappa = \{(x, y) : (\varphi(x), \psi(y)) \in X_1 \times Y_1\}.$$

There are relations between the operator multiplicity properties of κ_1 and those of κ.

Preservation properties – inverse images

Let $\varphi : X \to X_1$, $\psi : Y \to Y_1$ be measurable maps.
For $\kappa_1 \subseteq X_1 \times Y_1$ let

$$\kappa = \{(x, y) : (\varphi(x), \psi(y)) \in X_1 \times Y_1\}.$$

There are relations between the operator multiplicity properties of κ_1 and those of κ.

Roughly speaking,

- If κ_1 is an M-set (resp. an M_1-set) then κ is an M-set (resp. an M_1-set);

Preservation properties – inverse images

Let $\varphi : X \to X_1$, $\psi : Y \to Y_1$ be measurable maps.
For $\kappa_1 \subseteq X_1 \times Y_1$ let

$$\kappa = \{(x, y) : (\varphi(x), \psi(y)) \in X_1 \times Y_1\}.$$

There are relations between the operator multiplicity properties of κ_1 and those of κ.

Roughly speaking,

- If κ_1 is an M-set (resp. an M_1-set) then κ is an M-set (resp. an M_1-set);
- If κ_1 is not an M-set (resp. not an M_1-set) and φ, ψ are injective, then κ is not an M-set (resp. not an M_1-set)

An example

$f_i : X \to \mathbb{R}$, $g_i : Y \to \mathbb{R}$, $i = 1, \ldots, n$.

An example

$f_i : X \to \mathbb{R},\ g_i : Y \to \mathbb{R},\ i = 1, \ldots, n.$

$\kappa = \{(x, y) : f_i(x) \le g_i(y), i = 1, \ldots, n\}$

An example

$$f_i : X \to \mathbb{R},\ g_i : Y \to \mathbb{R},\ i = 1, \dots, n.$$

$$\kappa = \{(x, y) : f_i(x) \le g_i(y), i = 1, \dots, n\}$$

Theorem

Every compact operator in $\mathfrak{M}_{\max}(\kappa)$ is in the closure of the span of the rank one operators in $\mathfrak{M}_{\max}(\kappa)$.

An example

$$f_i : X \to \mathbb{R},\ g_i : Y \to \mathbb{R},\ i = 1, \ldots, n.$$

$$\kappa = \{(x, y) : f_i(x) \leq g_i(y), i = 1, \ldots, n\}$$

Theorem

Every compact operator in $\mathfrak{M}_{\max}(\kappa)$ is in the closure of the span of the rank one operators in $\mathfrak{M}_{\max}(\kappa)$.

The following are equivalent:

- κ is an M-set;
- κ is an M_1-set;
- κ has positive product measure;
- κ supports a non-trivial Hilbert-Schmidt operator;
- $\mathfrak{M}_{\max}(\kappa)$ contains a rank one operator.

Closable multipliers on group C*-algebras – the setting

Let $\psi : G \to \mathbb{C}$ be a measurable function.

Closable multipliers on group C*-algebras – the setting

Let $\psi : G \to \mathbb{C}$ be a measurable function.

It is well-known that pointwise multiplication on $L^1(G)$ by the function ψ defines a completely bounded map on $C_r^*(G)$ if and only if the function $N(\psi)$ is a Schur multiplier.

Let

$$D(\psi) = \{f \in L^1(G) : \psi f \in L^1(G)\};$$

it is easy to see that the operator $f \to \psi f$, $f \in D(\psi)$, viewed as a densely defined operator on $L^1(G)$, is closable.

Closable multipliers on group C*-algebras – the setting

Let $\psi : G \to \mathbb{C}$ be a measurable function.

It is well-known that pointwise multiplication on $L^1(G)$ by the function ψ defines a completely bounded map on $C_r^*(G)$ if and only if the function $N(\psi)$ is a Schur multiplier.

Let

$$D(\psi) = \{f \in L^1(G) : \psi f \in L^1(G)\};$$

it is easy to see that the operator $f \to \psi f$, $f \in D(\psi)$, viewed as a densely defined operator on $L^1(G)$, is closable.

Since $\lambda(L^1(G))$ is dense in $C_r^*(G)$ and $\|\lambda(f)\| \leq \|f\|_1$, $f \in L^1(G)$, the space $\lambda(D(\psi))$ is dense in $C_r^*(G)$ in the operator norm.

Closable multipliers on group C*-algebras – the setting

Let $\psi : G \to \mathbb{C}$ be a measurable function.

It is well-known that pointwise multiplication on $L^1(G)$ by the function ψ defines a completely bounded map on $C_r^*(G)$ if and only if the function $N(\psi)$ is a Schur multiplier.

Let

$$D(\psi) = \{f \in L^1(G) : \psi f \in L^1(G)\};$$

it is easy to see that the operator $f \to \psi f$, $f \in D(\psi)$, viewed as a densely defined operator on $L^1(G)$, is closable.

Since $\lambda(L^1(G))$ is dense in $C_r^*(G)$ and $\|\lambda(f)\| \leq \|f\|_1$, $f \in L^1(G)$, the space $\lambda(D(\psi))$ is dense in $C_r^*(G)$ in the operator norm.

Thus, the operator $S_\psi : \lambda(D(\psi)) \to C_r^*(G)$ given by $S_\psi(\lambda(f)) = \lambda(\psi f)$ is a densely defined operator on $C_r^*(G)$.

More on closable multipliers – some results

Theorem

Let G be a second countable locally compact group (satisfying a mild approximation property), $\psi : G \to \mathbb{C}$ be a measurable function and $\varphi = N(\psi)$. The following are equivalent:

- the operator S_ψ is closable;
- the operator S_φ is closable.

More on closable multipliers – some results

Theorem

Let G be a second countable locally compact group (satisfying a mild approximation property), $\psi : G \to \mathbb{C}$ be a measurable function and $\varphi = N(\psi)$. The following are equivalent:

- the operator S_ψ is closable;
- the operator S_φ is closable.

$E_\psi = \{t \in G : \psi$ does not almost belong to $A(G)$ at $t\}$.

More on closable multipliers – some results

Theorem

Let G be a second countable locally compact group (satisfying a mild approximation property), $\psi : G \to \mathbb{C}$ be a measurable function and $\varphi = N(\psi)$. The following are equivalent:

- the operator S_ψ is closable;
- the operator S_φ is closable.

$E_\psi = \{t \in G : \psi$ does not almost belong to $A(G)$ at $t\}$.

Theorem

Let G be a second countable locally compact group (satisfying the same approximation property), $\psi : G \to \mathbb{C}$ be a measurable function. Then

- if E_ψ is an U-set then S_ψ is closable;
- if E_ψ is an M_1-set then S_ψ is not closable.

p-versions

Let $\mathcal{S}_p$ be the Schatten p-class $(1 \leq p \leq \infty)$.

p-versions

Let $\mathcal{S}_p$ be the Schatten p-class $(1 \leq p \leq \infty)$.

$$\mathcal{S}_p(G) = \{T \in C_r^*(G) : M_a T M_b \in \mathcal{S}_p, \ \forall a, b \in C_c(G)\}.$$

p-versions

Let $\mathcal{S}_p$ be the Schatten p-class ($1 \leq p \leq \infty$).

$$\mathcal{S}_p(G) = \{T \in C^*_r(G) : M_a T M_b \in \mathcal{S}_p, \ \forall a, b \in C_c(G)\}.$$

Note that $\mathcal{S}_\infty(G) = C^*_r(G)$.

p-versions

Let $\mathcal{S}_p$ be the Schatten p-class ($1 \leq p \leq \infty$).

$$\mathcal{S}_p(G) = \{T \in C_r^*(G) : M_a T M_b \in \mathcal{S}_p, \ \forall a, b \in C_c(G)\}.$$

Note that $\mathcal{S}_\infty(G) = C_r^*(G)$.

For $T \in \mathcal{S}_p(G)$, let

$$\|T\|_{G,p} = \sup\{\|M_a T M_b\|_p : \|a\|_\infty \leq 1, \|b\|_\infty \leq 1\}.$$

Proposition

The space $\mathcal{S}_p(G)$ is complete, when equipped with $\|\cdot\|_{G,p}$. Moreover, if G is compact then $\mathcal{S}_p(G) = C_r^*(G) \cap \mathcal{S}_p$ and $\|\cdot\|_{G,p} = \|\cdot\|_p$.

p-versions

Definition

A closed subset $E \subseteq G$ will be called an *M^p-set* (or a *set of p-multiplicity*) if $J(E)^\perp \cap \mathcal{S}_p(G) \neq \{0\}$.

p-versions

Definition

A closed subset $E \subseteq G$ will be called an *M^p-set* (or a *set of p-multiplicity*) if $J(E)^{\perp} \cap \mathcal{S}_p(G) \neq \{0\}$.

Definition

An ω-closed subset $\kappa \subseteq X \times Y$ will be called an *operartor M^p-set* (or a *set of operator p-multiplicity*) if $\mathfrak{M}_{\max}(\kappa) \cap \mathcal{S}_p \neq \{0\}$.

Results about sets of p-multilicity

Theorem

Let G be compact. Then $E \subseteq G$ is a set of p-multiplicity if and only if E^* is a set of operator p-multiplicity.

Results about sets of p-multilicity

Theorem

Let G be compact. Then $E \subseteq G$ is a set of p-multiplicity if and only if E^* is a set of operator p-multiplicity.

For a general G, the same statement holds if $p = 1$ or $p = 2$.

Results about sets of p-multilicity

Theorem

Let G be compact. Then $E \subseteq G$ is a set of p-multiplicity if and only if E^* is a set of operator p-multiplicity.

For a general G, the same statement holds if $p = 1$ or $p = 2$.

Theorem

If $p \in \{1, 2\}$ and $E_i \subseteq G$ is not an M^p-set, $i = 1, 2$, then $E_1 \cup E_2$ is not an M^p-set.

Theorem

Let $p \in \{1, 2\}$ and $E_i \subseteq G_i$ be an M^p-set, $i = 1, 2$. Then $E_1 \times E_2 \subseteq G_1 \times G_2$ is an M^p-set.
Moreover, if G is compact the same conclusion holds for arbitrary $p \geq 1$.

THANK YOU VERY MUCH