Asymptotic Behaviour and Cyclic Properties of Tree-shift Operators

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29th July 2013, Gothenburg

Banach Algebras and Applications

Weighted unilateral shifts: $\{e_n\}_{n=0}^{\infty}$ is an ONB in $\mathcal{H}$, $\{w_n\}_{n=1}^{\infty} \subseteq \mathbb{C}$ bounded weight-sequence

$$W\colon \mathcal{H} \to \mathcal{H}, \quad We_n = w_{n+1}e_{n+1}.$$

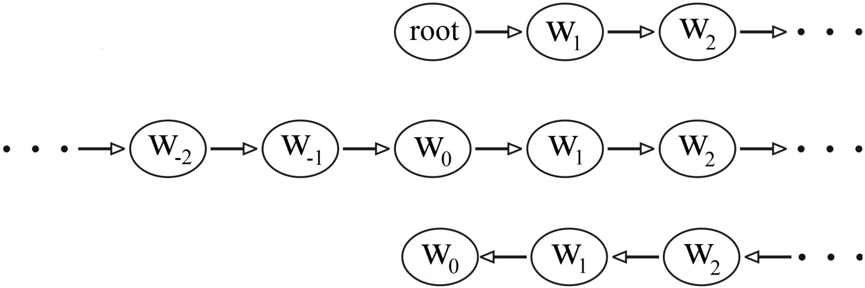
Weighted bilateral shifts: $\{e_n\}_{n=-\infty}^{\infty}$ is an ONB in $\mathcal{H}$, $\{w_n\}_{n=-\infty}^{\infty} \subseteq \mathbb{C}$ bounded weight-sequence

$$W\colon \mathcal{H} \to \mathcal{H}, \quad We_n = w_{n+1}e_{n+1}.$$

Weighted backward shifts: $\{e_n\}_{n=0}^{\infty}$ is an ONB in $\mathcal{H}$, $\{w_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ bounded weight-sequence

$$W\colon \mathcal{H} \to \mathcal{H}, \quad We_n = \begin{cases} w_{n-1}e_{n-1} & \text{if } n > 0, \\ 0 & \text{if } n = 0. \end{cases}$$

Graphs can be associated to these operators:



Sometimes we write the weights on the edges.

Why we investigate shift operators?

- Simple structure.
- They provide examples for many questions.
- After normal operators they are the best test operators.

A natural generalization: replace the previous graphs with directed trees and write some weights in the vertices which are not roots. $\longrightarrow$ This will naturally define a „tree-shift operator".

They were recently introduced in: **Z. J. Jablonski, I. B. Jung** and **J. Stochel**, *Weighted Shifts on Directed Trees*, Memoirs of the American Mathematical Society, Number 1017, 2012.

(hyponormality, co-hyponormality, subnormality and hyperexpansivity – simpler examples than before)

$\mathcal{T} = (V, E)$ is a directed graph, $V \equiv$ vertices, $E \equiv$ (directed) edges $E \subseteq V \times V \setminus \{(v, v) : v \in V\}$. $\mathcal{T}$ is a directed tree if

(i) $\mathcal{T}$ is connected, i.e.: $\forall\ u, v \in V, u \neq v\ \exists$ $u = v_0, v_1, \dots v_n = v \in V, n \in \mathbb{N}$ s. t. (v_{j-1}, v_j) or $(v_j, v_{j-1}) \in E$ for every $1 \leq j \leq n$.

(ii) For each vertex v there exists at most one other vertex u with the property that $(u, v) \in E$ (i.e. every vertex has at most one parent), and

(iii) $\mathcal{T}$ has no (directed) circuit, i.e.: $\nexists\ v_0, v_1, \dots v_n \in V, n \in \mathbb{N}$ distinct vertices s. t. $(v_{j-1}, v_j) \in E\ \forall\ 1 \leq j \leq n$ and $(v_n, v_0) \in E$.

If $(u, v) \in E$, then v is a child of u, u is the parent of v.
In notations: $v \in \mathrm{Chi}(u)$, $\mathrm{par}_{\mathcal{T}}(v) = \mathrm{par}(v) = u$.
$\mathrm{Chi}_{\mathcal{T}}(u) = \mathrm{Chi}(u) \equiv$ the set of all children of u.
If v has a parent, then it is unique.

$\mathrm{par}^k(v) := \underbrace{\mathrm{par}(\dots(\mathrm{par}}_{k\text{-times}}(v))\dots)$ if it makes sense,
and par^0 is the identity map.

If u has no parent $\longrightarrow$ root. If there exists one, then it is unique $u := \mathrm{root} = \mathrm{root}_{\mathcal{T}}$.

If $\mathrm{Chi}(u) = \emptyset \longrightarrow$ leaf. $\mathrm{Lea}(\mathcal{T}) \equiv$ the set of all leaves.

If $W \subseteq V$, then $\mathrm{Chi}(W) := \cup_{v \in W} \mathrm{Chi}(v)$,

$\mathrm{Gen}_{\mathcal{T}}(u) = \mathrm{Gen}(u) = \bigcup_{n=0}^{\infty} \left(\bigcup_{j=0}^{n} \mathrm{Chi}^j(\mathrm{par}^j(u)) \right)$ is the (whole) generation or the level of u.

Bounded Tree-shift Operator

$\ell^2(V)$: complex Hilbert space of all square summable functions.
The natural inner product: $\langle f, g \rangle = \sum_{u \in V} f(u)\overline{g(u)}$.
$e_u(v) = \delta_{u,v}$ $(u \in V)$, $\{e_u\}_{u \in V}$ ONB.
$W \subseteq V \longrightarrow \ell^2(W) = \vee\{e_v : v \in W\}$ which is a subspace (closed linear manifold).

Let $\underline{\lambda} = \{\lambda_v : v \in V \setminus \{\mathrm{root}\}\} \subseteq \mathbb{C}$ ($v \in V$ if $\mathcal{T}$ is rootless) be a set of weights $\sup\left\{\sqrt{\sum_{v \in \mathrm{Chi}(u)} |\lambda_v|^2} : u \in V\right\} < \infty$

$$\mathbf{S}_{\underline{\lambda}} : \ell^2(\mathbf{V}) \to \ell^2(\mathbf{V}), \quad \mathbf{e_u} \mapsto \sum_{\mathbf{v} \in \mathrm{Chi}(\mathbf{u})} \lambda_{\mathbf{v}} \mathbf{e_v},$$

$$\|S_{\underline{\lambda}}\| = \sup\left\{\sqrt{\sum_{v \in \mathrm{Chi}(u)} |\lambda_v|^2} : u \in V\right\}$$

$\mathcal{H}$: complex Hilbert space, $\mathcal{B}(\mathcal{H})$: bounded linear operators on it.
Suppose $T \in \mathcal{B}(\mathcal{H})$ is a contraction: $\|T\| \leq 1$,
then the following SOT-limits exist:

$$A = A_T = \lim_{n\to\infty} T^{*n}T^n \quad \text{and} \quad A_* = A_{T^*} = \lim_{n\to\infty} T^nT^{*n}.$$

A: the asymptotic limit of T, A_*: the asymptotic limit of T^*.

$h \in \mathcal{H}$ is a stable vector for T, if $\lim_{n\to\infty} \|T^nh\| = 0$. The set of all stable vectors: $\mathcal{H}_0 = \mathcal{H}_0(T) = \mathcal{N}(A_T)$.
$\mathcal{H}_0$ is a hyperinvariant subspace of T and will be called the stable subspace of T.
(a subspace is hyperinvariant for T if it is invariant for every C commuting with T)

A Classification of Contractions

If every vector is stable, then T is stable or of class $C_{0\cdot}$.
If $\mathcal{H}_0 = \{0\}$, then T is of class $C_{1\cdot}$.

If $T^* \in C_{i\cdot}(\mathcal{H})$ ($i = 0$ or 1), then T is of class $C_{\cdot i}$.
$C_{ij}(\mathcal{H}) = C_{i\cdot}(\mathcal{H}) \cap C_{\cdot j}(\mathcal{H})$.

Because $\mathcal{H}_0(T)$ or $\mathcal{H}_0(T^*)^\perp$ is hyperinvariant, if $T \notin C_{00} \cup C_{10} \cup C_{01} \cup C_{11}$, then it has a non-trivial hyperinvariant subspace (C_{11} can be also handled).
For C_{11} contractions there is a nice structure result (**C. Foias** and **B. Sz.-Nagy**)
So the hyper-invariant subspace problem is only open in the classes C_{00}, C_{10}, C_{01}.

$X \in \mathcal{B}(\mathcal{H}, \mathcal{R}(A_T)^-), Xh = A_T^{1/2}h.$
There exists a unique isometry $U \in \mathcal{B}(\mathcal{R}(A_T)^-)$ s. t.

$$XT = UX.$$

The pair (X, U) is a canonical realization of the so called isometric asymptote of T.
(There is a more general definition, but we will not use it)
(Important because of the (hyper)invariant subspace problem)

Application 1 (C. Foias and B. Sz.-Nagy):

With the isometric asymptote we can prove that a contraction T from the class C_{11} are quasi-similar to a unitary operator.
$T \sim U$ (quasi-similar), if $\exists\, X, Y \in \mathcal{B}(\mathcal{H})$ with dense range and trivial kernel s. t. $XT = UX$ and $YU = TY$.
(If they are similar, then $Y = X^{-1}$).

$h \in \mathcal{H}$ is a cyclic vector for $T \in \mathcal{B}(\mathcal{H})$

$$\vee\{T^n h \colon n \in \mathbb{Z}_+\} = \mathcal{H}.$$

Then T is cyclic operator.

Application 2:

If U is not cyclic and $T \in C_{1\cdot}(\mathcal{H}) \implies T$ is also non-cyclic.

If U^* is cyclic and $T \in C_{1\cdot}(\mathcal{H}) \implies T^*$ is also cyclic.

A consequence: Every contractive $C_{\cdot 1}$ weighted bilateral shift is cyclic.

Application 3:

T is similar to an isometry $\iff$ A_T is invertible.
T is similar to a unitary $\iff$ both A and A_* are invertible.

W.L.o.G. we can and we will assume in the talk that all weights are **strictly positive** and $\mathcal{H}$ is **separable** ($|V| \leq \aleph_0$).

Theorem

Let $S_{\underline{\lambda}}$ be a tree-shift contraction. Then the asymptotic limit A is a positive operator s. t.

$$Ae_u = \alpha_u e_u \quad \forall\ u \in V,$$

where $\alpha_u = \lim_{n\to\infty} \sum_{v\in\mathrm{Chi}^n(u)} \prod_{j=0}^{n-1} \lambda^2_{\mathrm{par}^j(v)}$.

Corollary

If $V' = \{v' \in V : e_{v'} \notin \mathcal{H}_0\}$, *then* $\mathcal{H}_0 = \ell^2(V \setminus V')$ *and* $\mathcal{H}_0^{\perp} = \ell^2(V')$.

Proposition

The followings are valid for every vertex $u \in V$:

(i) $u \notin V' \Longrightarrow \mathrm{Chi}(u) \subseteq V \setminus V'$,

(ii) $u \notin V' \iff \mathrm{Chi}(u) \subseteq V \setminus V'$; *this is fulfilled in the special case when* u *is a leaf,*

(iii) $u \in V' \Longrightarrow \mathrm{par}^k(u) \in V', \ \forall\, k \in \mathbb{Z}_+$,

(iv) $\mathcal{T}' = (V', E' = E \cap (V' \times V'))$ *is a leafless subtree,*

(v) *if* $\mathcal{T}$ *has no root, neither has* $\mathcal{T}'$, *and*

(vi) *if* $\mathcal{T}$ *has a root, then either* $S_{\underline{\lambda}} \in C_{0\cdot}(\ell^2(V))$ *or* $\mathrm{root}_{\mathcal{T}} = \mathrm{root}_{\mathcal{T}'}$.

Theorem

If $S_{\underline{\lambda}}$ is a tree-shift contraction, then

(i) *If $\mathcal{T}$ has a root, then $S_{\underline{\lambda}} \in C_{\cdot 0}(\ell^2(V))$.*

(ii) *If $\mathcal{T}$ is rootless, then $\mathcal{H}_0(S_{\underline{\lambda}}^*)^\perp = \vee\{h_u \colon u \in V\}$ where*

$$h_u = \sum_{v \in \mathrm{Gen}(u)} \prod_{j=0}^{\infty} \lambda_{\mathrm{par}^j(v)} \cdot e_v \in \ell^2(V).$$

$h_u = h_v \iff v \in \mathrm{Gen}(u)$ *and* h_u *are eigen-vectors:*
$A_* h_u = a_u h_u \quad \forall\, u \in V$ *with the corresponding eigen-values*

$$a_u = \|h_u\|^2 = \sum_{v \in \mathrm{Gen}(u)} \prod_{j=0}^{\infty} \lambda^2_{\mathrm{par}^j(v)}.$$

So, every level has one such h_u. Moreover, if h_u is not zero for a vertex u, then it is not zero for every $u \in V$.

$\mathrm{Br}(\mathcal{T}) = \sum_{\substack{u \in V \\ u \notin \mathrm{Lea}(\mathcal{T})}} (|\mathrm{Chi}(u)| - 1)$ is the branching index of $\mathcal{T}$.

S^+ is the simple unilateral shift (all weights are 1) and S is the simple bilateral shift. They can be represented as multiplication operators by $\chi(z) = z$ on $H^2(\mathbb{T})$ and $L^2(\mathbb{T})$, respectively.

The isometry $T \in \mathcal{B}(\mathcal{H})$ is called completely non-unitary (c.n.u. for short) if the only reducing subspace on which T acts as a unitary operator is the trivial one: $\{0\}$.
An isometry is c.n.u iff it is an orthogonal sum of simple unilateral shifts.

Theorem

For a non-$C_{0\cdot}$ tree-shift contraction $S_{\underline{\lambda}} \in \ell^2(V)$, the isometric asymptote $U \in \mathcal{B}(\ell^2(V'))$ is unitarily equivalent to:

(i) $\sum_{j=1}^{\mathrm{Br}(\mathcal{T}')+1} \oplus S^+$, *if $\mathcal{T}$ has a root,*

(ii) $\sum_{j=1}^{\mathrm{Br}(\mathcal{T}')} \oplus S^+$, *if $\mathcal{T}$ has no root and U is a c.n.u. isometry, i.e.: $\sum_{v' \in \mathrm{Gen}_{\mathcal{T}'}(u')} \prod_{j=0}^{\infty} \beta^2_{\mathrm{par}^j(v')} = 0$ for some (and then for every) $u' \in V'$,*

(iii) $S \oplus \sum_{j=1}^{\mathrm{Br}(\mathcal{T}')} \oplus S^+$, *if $\mathcal{T}$ has no root and U is not a c.n.u. isometry.*

Theorem

Suppose that the contractive tree-shift operator $S_{\underline{\lambda}}$ is not in the class $C_{\cdot 0}$. Then $\mathcal{T}$ has no root and the isometry U_* acts as follows:

$$U_* h_u = \frac{\sqrt{a_u}}{\sqrt{a_{\mathrm{par}(u)}}} \cdot h_{\mathrm{par}(u)},$$

where $h_u \neq 0$ for every $u \in V$. In fact U_* is a simple unilateral shift, if $\mathrm{Chi}(\mathrm{Gen}(u)) = \emptyset$ for some $u \in V$, and a simple bilateral shift elsewhere.

Corollary

Consider the tree-shift contraction $S_{\underline{\lambda}}$. *Then the followings hold*

(i) $S_{\underline{\lambda}}$ *is similar to an isometry if and only if* $\inf\{\alpha_u : u \in V\} > 0$.

(ii) $S_{\underline{\lambda}}$ *is similar to a co-isometry if and only if it is a bilateral weighted shift with* $\prod_{j=-\infty}^{\infty} \lambda_j > 0$, *or it is a unilateral weighted backward shift with* $\prod_{j=0}^{\infty} \lambda_j > 0$. *Then it is similar to the simple bilateral or the simple backward shift operator, respectively.*

Easy to see: If $\mathrm{co}-\dim(\mathcal{R}(T)^{-}) > 1 \Longrightarrow T$ has no cyclic vectors.

$$\Downarrow$$

- if $\mathcal{T}$ has a root and $\mathrm{Br}(\mathcal{T}) > 0 \Longrightarrow S_{\underline{\lambda}}$ has no cyclic vectors,
- if $\mathcal{T}$ is rootless and $\mathrm{Br}(\mathcal{T}) > 1 \Longrightarrow S_{\underline{\lambda}}$ has no cyclic vectors.

The pure tree-shift case is when $\mathrm{Br}(\mathcal{T}) = 1$ and $\mathcal{T}$ has no root.

$\mathrm{Br}(\mathcal{T}) = 0 \implies S_{\underline{\lambda}}$ is

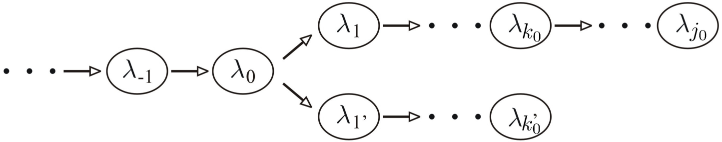
- a weighted bilateral shift (no characterization for cyclicity is known) (for supercyclicity, hypercyclicity . . . there are characterizations),
- a weighted unilateral shift (easy: always cyclic),
- a weighted backward shift (always cyclic, later),
- a cyclic nilpotent operator acting on a finite dimensional space.

There are cyclic and non-cyclic bilateral shifts, see:
A. L. Shields, Weighted shift operators and analytic function theory, *Topics in Operator Theory*, Math. Surveys 13, Amer. Math. Soc., Providence, R. I., 1974, 49–128.

Theorem

Suppose that B is a weighted backward shift of countable multiplicity. Then B is cyclic if and only if there is at most one zero weight.

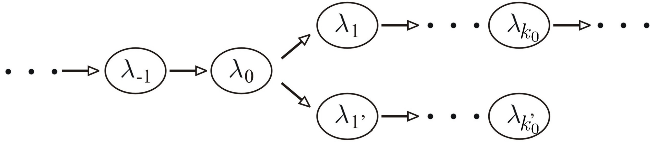
In case the multiplicity is 1 the was obtained by **Z. Guang Hua** in 1984 (written in Chinese).



Theorem

If the directed tree $\mathcal{T}$ has no root, $\mathrm{Br}(\mathcal{T}) = 1$ and have 2 leaves, then every bounded tree-shift operator on it is cyclic.

Proof. $S_{\underline{\lambda}}$ is similar to a backward shift, which has at most one zero weight. ■

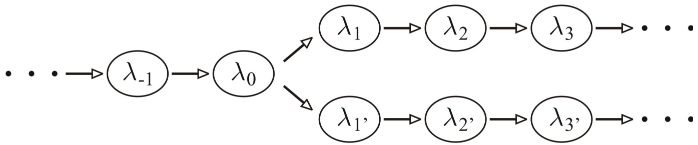


Theorem

Suppose $\mathcal{T}$ has a unique leaf. A tree-shift operator $S_{\underline{\lambda}}$ on $\mathcal{T}$ is cyclic if and only if the bilateral shift W with weights $\{\lambda_n\}_{n=-\infty}^{\infty}$ is cyclic. In particular, if $S_{\underline{\lambda}} \notin C_{\cdot 0}(\ell^2(V))$, then $S_{\underline{\lambda}}$ is cyclic.

Proposition

The operator $S \oplus S^+$ has no cyclic vectors.



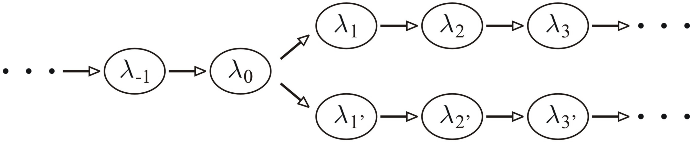
Theorem

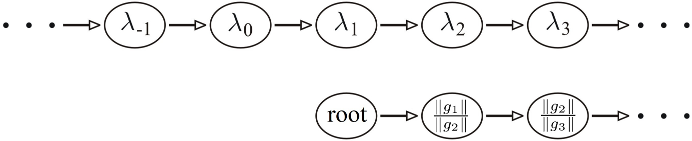
Suppose that $\mathcal{T}$ is rootless and $\mathrm{Br}(\mathcal{T}) = 1$. If the tree-shift contraction $S_{\underline{\lambda}}$ is of class $C_{1\cdot}$, then it has no cyclic vectors.

Proof. The isometric asymptote of $S_{\underline{\lambda}}$ is unitarily equivalent to $S \oplus S^+$. ■

On this $\mathcal{T}$ the contrary may also happen.

The graphs of $S_{\underline{\lambda}}$ and W





$$g_k = \prod_{j=1}^{k} \frac{1}{\lambda_j} \cdot e_k - \prod_{j=1}^{k} \frac{1}{\lambda_{j'}} \cdot e_{k'} \qquad (k \in \mathbb{N}) .$$

Theorem

If $\left\{ \prod_{j=1}^{k} \frac{\lambda_{j'}}{\lambda_j} : k \in \mathbb{N} \right\}$ *is bounded, then* $S_{\underline{\lambda}}$ *is similar to* W*.*

Corollary

If $S_{\underline{\lambda}} \notin C_{0\cdot}(\ell^2(V))$*, then it is similar to* W*.*

Theorem

There is a tree-shift operator on the previous directed tree which is cyclic.

$$S_k^+ := \underbrace{S^+ \oplus \cdots \oplus S^+}_{k \text{ times}} \ (k \in \mathbb{N}),$$

$$S_{\aleph_0}^+ := \underbrace{S^+ \oplus S^+ \oplus \dots}_{\aleph_0 \text{ many}}.$$

Theorem

The operator $S \oplus (S_k^+)^*$ is cyclic for every $k \in \mathbb{N}$.

Theorem

The followings are valid:

- (i) *If $\mathcal{T}$ has a root and the tree-shift contraction $S_{\underline{\lambda}}$ on it is of class $C_{1\cdot}$, then $S_{\underline{\lambda}}^*$ is cyclic.*
- (ii) *If $\mathcal{T}$ is rootless, $\mathrm{Br}(\mathcal{T}) < \infty$ and the tree-shift contraction $S_{\underline{\lambda}}$ on it is of class $C_{1\cdot}$, then $S_{\underline{\lambda}}^*$ is cyclic.*

Question

Is the operator $S \oplus (S^{+}_{\aleph_0})^*$ *cyclic?*

This research was realized in the frames of TÁMOP 4.2.4. A/2-11-1-2012-0001 ”National Excellence Program - Elaborating and operating an inland student and researcher personal support system”. The project was subsidized by the European Union and co-financed by the European Social Fund

Thank You For Your Kind Attention!