# Crossed products by arbitrary endomorphisms

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Check out also my poster
on Wednesday !!!

B. K. Kwaśniewski, A. V. Lebedev "Crossed products by endomorphisms and reduction of relations in relative Cuntz-Pimsner algebras"
J. of Funct. Analysis, 264 (2013), no. 8, 1806-1847

# Question of the Day: What is crossed product?

Throughout $A$ is unital $C^*$-algebra.

Crossed product by an automorphism $\alpha : A \to A$ is a universal $C^*$-algebra $C^*(A, u)$ generated by $A$ and $u$ subject to relations:

$$\alpha(a) = uau^*, \qquad \alpha^{-1}(a) = u^*au, \qquad a \in A$$

Problem If $\alpha : A \to A$ is an endomorphism, then ~~$\alpha^{-1}(a) = u^*au$~~.

**What relation should we use instead?**

Let $A \subset B$ be $C^*$-algebras with a common unit 1, $U \in B$.

**Proposition (the Hint).**

Let $\alpha : A \to A$ be a map of the form $\alpha(a) = UaU^*$. Then

$$\alpha \text{ is an endomorphism } \iff U \text{ partial isometry, } U^*U \in A',$$

where $A'$ is the commutant of $A$.

# Towards the crossed product construction

Assume $\alpha(a) = UaU^*$ is an endomorphism of $A$.

**Proposition.**

**1)**

$$C^*(A, U) = \overline{\text{span}}\{U^{*n} a U^m : a \in A,\ \ n, m \in \mathbb{N}\}$$

**2)**

$$J = \{a \in A : U^*Ua = a\} = U^*UA \cap A$$

is an ideal in $A$ such that $J \cap \ker \alpha = \{0\}$ $(J \subset (\ker \alpha)^{\perp})$

**Rem.** In the crossed product construction 

 the elements $U^{*n} a U^m$ are the 'bricks'

 the ideal $J = \{a \in A : U^*Ua = a\}$ is the 'cement'

# 1) Reading off the algebraic structure from $C^*(A, U)$

Consider infinite matrices with entries labeled by $\mathbb{N} = \{0, 1, 2, ...\}$

$$\mathcal{M}(A) := \{[a_{n,m}] : a_{n,m} \in A \text{ only finite entries non zero}\}$$

and a mapping $\Psi : \mathcal{M}(A) \to C^*(A, U)$ given by

$$\Psi([a_{n,m}]) = \sum_{n,m\in\mathbb{N}} U^{*n} a_{n,m} U^m$$

**Proposition.** The map $\Psi$ becomes a $^*$-homomorphism if

we define on $\mathcal{M}(A)$ the $^*$-algebra structure $(\mathcal{M}(A), +, \cdot, *, \star)$ as follows

$$(a + b)_{m,n} := a_{m,n} + b_{m,n}, \tag{1}$$

$$(\lambda a)_{m,n} := \lambda a_{m,n} \tag{2}$$

$$(a^*)_{m,n} := a^*_{n,m} \tag{3}$$

and

$$a \star b := a \cdot \sum_{j=0}^{\infty} \Lambda^j(b) + \sum_{j=1}^{\infty} \Lambda^j(a) \cdot b \tag{4}$$

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$$a \star b = a \cdot \sum_{j=0}^{\infty} \Lambda^j(b) + \sum_{j=1}^{\infty} \Lambda^j(a) \cdot b \tag{4}$$

where $\cdot$ is matrix multiplication and $\Lambda : \mathcal{M}(A) \to \mathcal{M}(A)$ is given by

$$\Lambda([a_{n,m}]) := \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & \alpha(a_{0,0}) & \alpha(a_{0,1}) & \alpha(a_{0,2}) & \cdots \\ 0 & \alpha(a_{1,0}) & \alpha(a_{1,1}) & \alpha(a_{1,2}) & \cdots \\ 0 & \alpha(a_{2,0}) & \alpha(a_{2,1}) & \alpha(a_{2,2}) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

# 2) Calculation of norm of elements in $C^*(A,U)$

An element $[a_{n,m}] \in \mathcal{M}(A)$ is *k-diagonal*, where $k \in \mathbb{Z}$, if it is of the form

$$\begin{matrix} k\left\{ \right. \\ r+1\left\{ \right. \end{matrix} \begin{pmatrix} & & 0 \\ a_{k,0} & & \\ & \ddots & \\ 0 & & a_{r+k,r} \\ & & \end{pmatrix} \quad \text{if} \quad k \geq 0, \text{ or } \quad \begin{pmatrix} \overbrace{\phantom{00}}^{-k} & \overbrace{a_{0,-k} \quad\quad}^{r+k+1} & 0 \\ & \ddots & \\ & & a_{r+k,r} \\ 0 & & \end{pmatrix}, \text{ if } \quad k < 0.$$

**Proposition.** If $a = \Psi([a_{n,m}])$ where $[a_{n,m}]$ is $k$-diagonal, then

$$\|a\| = \lim_{n\to\infty} \max \left\{ \max_{i=1,\dots,n} \left\{ d\Big( \sum_{\substack{j=0,\\ j+k\geq 0}}^{i} \alpha^{i-j}(a_{j+k,j}), J\Big) \right\}, \ d(a_{n+k,n}, \ker\alpha) \right\} \qquad (5)$$

where $J = \{a \in A : U^*Ua = a\}$ and $d(a,I) = \inf_{b\in I} \|a-b\|$.

# Theorem (crossed product construction)

Let $\alpha : A \to A$ be an endomorphism and $J \subset (\ker\alpha)^{\perp}$. There is a unique $C^*$-seminorm $\|\cdot\|$ on the $^*$-algebra $(\mathcal{M}(A), +, \cdot, *, \star)$ such that (5) holds and either

1) $$\|\sum_k a_k\| = \|\sum_k \lambda^k a_k\|, \qquad \textit{for all } \lambda \in \mathbb{T},$$

or

2) $$\|a_0\| \leq \|\sum_k a_k\|$$

for all $k$-diagonal elements $a_k \in \mathcal{M}(A)$. This $C^*$-seminorm yields a $C^*$-algebra

$$C^*(A, \alpha, J) := \overline{\mathcal{M}(A)/\|\cdot\|},$$

which is generated by the elements

$$u := \begin{pmatrix} 0 & \alpha(1) & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad \widehat{a} := \begin{pmatrix} a & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad a \in A,$$

and universal subject to relations: $\widehat{\alpha(a)} = u\widehat{a}u^*$, $J = \{a \in A : u^*u\widehat{a} = \widehat{a}\}$.

**Definition** Let $\alpha : A \to A$ be an endomorphism and $J \subset (\ker \alpha)^{\perp}$.

We call $C^*(A, \alpha, J)$ the *crossed product of $A$ by $\alpha$ relative to $J$*.
If $J = (\ker \alpha)^{\perp}$ we write $C^*(A, \alpha)$ and call it *crossed product of $A$ by $\alpha$*.

$C^*(A, \alpha, J)$ is a universal $C^*$-algebra generated by $A$ and $u$ subject to

$$\alpha(a) = uau^*, \; a \in A, \qquad J = \{a \in A : u^*ua = a\} = u^*uA \cap A.$$

**Proposition**

$\ker \alpha$ unital $\Longrightarrow C^*(A, \alpha)$ is universal generated by $A$ and $u$ subject to

$$\alpha(a) = uau^*, \; a \in A, \qquad u^*u \in A.$$

$\alpha$ monomorphism $\Longrightarrow C^*(A, \alpha) = A \rtimes_\alpha \mathbb{N}$ Stacey's crossed product:

$$\alpha(a) = uau^*, \; a \in A, \qquad u^*u = 1.$$

$\alpha$ automorphism $\Longrightarrow C^*(A, \alpha) = A \rtimes_\alpha \mathbb{Z}$ classical crossed product:

$$\alpha(a) = uau^*, \qquad \alpha^{-1}(a) = u^*au, \qquad a \in A.$$

# Relevant constructions

## $J$-Reduction of an endomorphism

Suppose $J$ is an arbitrary ideal in $A$. Let $J_\infty$ be the smallest $\alpha$-invariant ideal s.t. putting $q : A \to A/J_\infty$ and

$$A_r := q(A), \ \alpha_r \circ q := q \circ \alpha, \ J_r := q(J).$$

we have $J_r \subset (\ker \alpha_r)^\perp$.
$C^*(A_r, \alpha_r, J_r)$ is generated by an image of $A$ and $u$ subject to relations

$$\alpha(a) = uau^*, \quad a \in A, \quad J \subset u^*uA \cap A$$

## $J$-Unitization of kernel

If $J \subset (\ker \alpha)^\perp$ one can construct an endomorphism $\alpha_J : A_J \to A_J$ such that

1) $A \subset A_J$, $\alpha_J|_A = \alpha$, $\ker \alpha_J$ is unital

2) $A_J = A \iff \begin{pmatrix} \ker \alpha \text{ is unital} \\ J = (\ker \alpha)^\perp \end{pmatrix}$

3) $C^*(A, \alpha, J) \cong C^*(A_J, \alpha_J)$

$$\alpha_J(a) = uau^*, \quad a \in A_J, \quad u^*u \in A_J$$

**Hereditation of range.** Suppose $\ker \alpha$ is unital.

There is an endomorphism $\beta : B \to B$ extending $\alpha : A \to A$ such that $\ker \beta$ is **unital**, $\beta(B)$ is **hereditary** and $C^*(A, \alpha) \cong C^*(B, \beta)$.

# Remarks on Exel's crossed product and topological freeness

Standing assumptions: $\ker\alpha$ is unital and $\alpha(A)$ is hereditary in $A$.

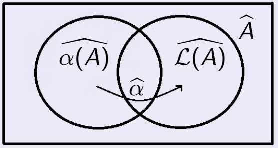
**Proposition [Kwa$_1$, KL, ABL]**

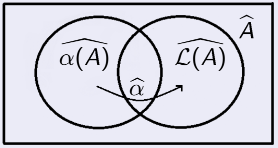
There is a unique non-degenerate transfer operator $\mathcal{L}: A \to A$ for $\alpha: A \to A$ and $C^*(A, \alpha)$ is a universal $C^*$-algebra generated by $A$ and $u$ subject to:

$$\alpha(a) = uau^*, \qquad \mathcal{L}(a) = u^*au, \qquad a \in A.$$

Moreover, $C^*(A, \alpha) \cong A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ – Exel's crossed product.

**Remark.** $\alpha : \mathcal{L}(A) \to \alpha(A)$ is an isomorphism and its dual $\widehat{\alpha} : \widehat{\alpha(A)} \to \widehat{\mathcal{L}(A)}$ may be treated as a partial homeomorphism of $\widehat{A}$:




**Uniqueness Theorem [Kwa$_2$].** If $\widehat{\alpha}$ is *topologically free* (the set of periodic points of period $n \in \mathbb{N}$ has empty interior), then for any faithful representation $\pi : A \to B$ and $U \in B$ such that

$$\pi(\alpha(a)) = U\pi(a)U^*, \qquad \pi(\mathcal{L}(a)) = U^*\pi(a)U, \qquad a \in A,$$

the mappings

$$a \mapsto \pi(a), \;\; a \in A, \qquad u \mapsto U$$

yield the isomorphism $C^*(A, \alpha) \cong C^*(\pi(A), U)$.

**Open problem:**

How to define ☺ and topological freeness for an arbitrary endomorphism?

# References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