Uniformly square Banach spaces
(joint work with J. Langemets and V. Lima)

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Banach Algebras and Applications
dedicated to the memory of William G. Bade
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Notation

- X will denote a Banach space and X^* its dual.
- B_X the unit ball, B_X^o the open unit ball of X.
- S_X the unit sphere of X.

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Basic definitions

Definition (Schäffer, 1976)

A point $x \in S_X$ is called uniformly non-square if there exists $\delta > 0$ such that

$$\max \|x \pm y\| \geq 1 + \delta \text{ for all } y \in S_X.$$

- If $x \in S_X$, then

$$2 \leq \|x + y\| + \|x - y\| \leq 2 \max \|x \pm y\| \text{ for all } y \in X$$

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- So $x \in S_X$ is NOT uniformly non-square

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there is $(y_n) \subset S_X$ with $\|x \pm y_n\| \to 1$.

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Definition

X is

a) locally uniformly square (LUS) if for every $x \in S_X$ there is a sequence $(y_n) \subset S_X$ such that $\|x \pm y_n\| \to 1$.

b) weakly uniformly square (ωUS) if X is LUS and $y_n \to 0$ weakly.

c) uniformly square (US) if for every $(x_i)_{i=1}^N \subset S_X$ there is a sequence $(y_n) \subset S_X$ such that $\|x_i \pm y_n\| \to 1$ for every $i = 1, \ldots, N$.

- Obviously b) $\Rightarrow$ a) and c) $\Rightarrow$ a).
- We will show: c) $\Rightarrow$ b).

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The space of null-sequences - c_0

- Let $x = (x_k) \in S_{c_0}$. Then $\|x \pm e_n\| \to 1$ where e_n is the n'th canonical unit vector in c_0. So c_0 is *LUS*,
- ... and ωUS as $e_n \to 0$ weakly.
- ... and *US* by the same idea as for the *LUS* case.
- Actually: $c_0(X_i)$ is *US* for any sequence of spaces X_i.

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- Let $x = (1, 1, \ldots, 1, \ldots) \in S_c$. Now, if $\|x \pm y_n\| \to 1$, then $\|y_n\| \neq 1$. Because: if the value of one term of y_n were close to ± 1, then the maximum of that term of $x \pm y_n$ would be close to 2. So c is not *LUS*.
- Actually $\|y_n\| \to 0$. Reason: $x = (1, 1, \ldots, 1, \ldots)$ is a strong extreme point in S_c.

Fact

The unit ball of LUS spaces cannot have strong extreme points.

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Fact

The unit ball of LUS spaces cannot have strong extreme points.

More examples
not containing c_0

- $L_1[0,1]$ is ωUS, but not US.

Theorem (Kubiak, to appear)

The Cesaro function spaces C_p, $1 \le p < \infty$ are ωUS.

For $1 \le p < \infty$, $C_p = \{f \in L_p : \int_0^1 (1/x \int_0^x |f(t)| \mathrm{d}t)^p < \infty\}$ with norm $\|f\| = (\int_0^1 (1/x \int_0^x |f(t)| \mathrm{d}t)^p)^{1/p}$.

- Note that C_p is strictly convex.

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Even more examples

M-embedded spaces

- X is called M-embedded if $X^{***} = X^* \oplus_1 X^\perp$.

Theorem

Non-reflexive M-embedded spaces are US.

- In particular: $c_0(\Gamma)$, $K(H)$, $K(\ell_p, \ell_q)$ where $1 < p \leq q < \infty$ are *US*.
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A Banach space X has the

i) *local diameter 2 property (LD2P)* if every slice of B_X has diameter 2.

ii) *diameter 2 property (D2P)* if every non-empty relatively weakly open subset of B_X has diameter 2.

iii) *strong diameter 2 property (SD2P)* if every finite convex combination of slices of B_X has diameter 2.

Theorem (Becerra Guerrero, López-Pérez, Rueda Zoca)

LD2P $\not\Rightarrow$ *D2P*.

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$*$*US* spaces are diameter 2 spaces

Proposition (Kubiak, to appear)

- *LUS* $\Rightarrow$ *LD2P*.
- ω*US* $\Rightarrow$ *D2P*.

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$LUS \Rightarrow LD2P$

Sketch of proof.

Let $\varepsilon > 0$, $x^* \in S_{X^*}$, and $S = S(x^*, \varepsilon) = \{x \in B_X : x^*(x) > 1 - \varepsilon\}$ be a slice of B_X. Choose $x \in S \cap S_X$ and find a sequence $(y_n) \subset S_X$ such that $\|x \pm y_n\| \to 1$. Then $x \pm y_n \in S$ for large ns, and then we get

$$\|x + y_n - (x - y_n)\| = 2\|y_n\| = 2.$$

□

- Similarly $\omega US \Rightarrow D2P$ and $US \Rightarrow SD2P$.

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A characterization of *LUS* spaces

Theorem

For a Banach space X. TFAE.

i) *X is LUS.*

ii) *For every $\varepsilon > 0$ and every 1-dimensional subspace $F \subset X$ there exists $y \in S_X$ such that*

$$(1-\varepsilon)\max\{\|x\|, |\lambda|\} \leq \|x + \lambda y\| \leq (1+\varepsilon)\max\{\|x\|, |\lambda|\}$$

for all $x \in F$ and $\lambda \in \mathbb{R}$.

Corollary

LUS spaces contain almost isometric copies of ℓ^2_∞.

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LUS spaces contain almost isometric copies of ℓ^2_∞.

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$US \Rightarrow \omega US$

Proposition

If X is US, then it is ωUS.

Sketch of proof.

Choose $(x_i)_{i=1}^N \subset S_F$. Then $\|(x_i, \pm e_n)\| = 1$ in $F \oplus_\infty c_0$. Define $S : F \oplus_\infty c_0 \to \text{span}\,(F, Y)$ by $S(f, x) = f + T(x)$ where $T : c_0 \to Y$ is the ε-isometry. Then $\|S(x_i, \pm e_n)\| = \|x_i \pm T(e_n)\| \to 1$, and $T(e_n) \to 0$ weakly in X as $e_n \to 0$ weakly in c_0. $\square$

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Definition (Behrends and Harmand)

X has the intersection property (IP) if for all $\varepsilon > 0$ there exist $(x_i)_{i=1}^N \subset B_X{}^o$ such that $\|y\| \leq \varepsilon$ if $\|x_i \pm y\| \leq 1$ for every $i = 1, \ldots, N$, i.e. the intersection of the balls $B(\pm x_i, 1)$ is contained in $B(0, \varepsilon)$.

If X fails the IP then there exists $\varepsilon > 0$ such that for all $(x_i)_{i=1}^N \subset B_X^o$ there is y with $\|y\| > \varepsilon$ and $\|x_i \pm y\| \leq 1$ for every $i = 1, \ldots, N$.

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$US \Rightarrow$ fails the IP. But the converse is not true.

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$$X = \{f \in C[0,1] : f(0) = 2f(1)\}.$$

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