Uniformly square Banach spaces
(joint work with J. Langemets and V. Lima)

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Banach Algebras and Applications
dedicated to the memory of William G. Bade
Gothenburg
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**Notation**

- $X$ will denote a Banach space and $X^*$ its dual.
- $B_X$ the unit ball, $B_X^o$ the open unit ball of $X$.
- $S_X$ the unit sphere of $X$. 

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Basic definitions

**Definition (Schäffer, 1976)**

A point \( x \in S_X \) is called uniformly non-square if there exists \( \delta > 0 \) such that

\[
\max \| x \pm y \| \geq 1 + \delta \quad \text{for all} \quad y \in S_X.
\]

- If \( x \in S_X \), then
  \[
  2 \leq \| x + y \| + \| x - y \| \leq 2 \max \| x \pm y \| \quad \text{for all} \quad y \in X
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- So \( x \in S_X \) is NOT uniformly non-square

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c) uniformly square (\( US \)) if for every \( (x_i)_{i=1}^N \subset S_X \) there is a sequence \( (y_n) \subset S_X \) such that \( \|x_i \pm y_n\| \to 1 \) for every \( i = 1, \ldots, N \).

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We will show: c) \( \Rightarrow \) b).
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- Let \( x = (1, 1, \ldots, 1, \ldots) \in S_c \). Now, if \( \|x \pm y_n\| \to 1 \), then \( \|y_n\| \neq 1 \). Because: if the value of one term of \( y_n \) were close to \( \pm 1 \), then the maximum of that term of \( x \pm y_n \) would be close to 2. So \( c \) is not \( LUS \).

- Actually \( \|y_n\| \to 0 \). Reason: \( x = (1, 1, \ldots, 1, \ldots) \) is a strong extreme point in \( S_c \).

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*The unit ball of LUS spaces cannot have strong extreme points.*
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More examples not containing $c_0$

- $L_1[0, 1]$ is $\omega US$, but not $US$.

**Theorem (Kubiak, to appear)**

The Cesaro function spaces $C_p$, $1 \leq p < \infty$ are $\omega US$.

For $1 \leq p < \infty$, $C_p = \{ f \in L_p : \int_0^1 (1/x \int_0^x |f(t)|dt)^p < \infty \}$

with norm $\|f\| = (\int_0^1 (1/x \int_0^x |f(t)|dt)^p)^{1/p}$.

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X is called M-embedded if $X^{***} = X^* \oplus_1 X^\perp$.

**Theorem**

Non-reflexive M-embedded spaces are US.

- In particular: $c_0(\Gamma)$, $K(H)$, $K(\ell_p, \ell_q)$ where $1 < p \leq q < \infty$ are US.
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i) *local diameter 2 property (LD2P)* if every slice of $B_X$ has diameter 2.

ii) *diameter 2 property (D2P)* if every non-empty relatively weakly open subset of $B_X$ has diameter 2.

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Let $\varepsilon > 0$, $x^* \in S_{X^*}$, and

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be a slice of $B_X$. Choose $x \in S \cap B_X$ and find a sequence $(y_n) \subset S_X$ such that $\|x \pm y_n\| \to 1$. Then $x \pm y_n \in S$ for large $n$s, and then we get

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Similarly \( \omega US \Rightarrow D2P \) and \( US \Rightarrow SD2P \). 
**Sketch of proof.**

Let \( \varepsilon > 0 \), \( x^* \in S_{X^*} \), and  
\[
S = S(x^*, \varepsilon) = \{ x \in B_X : x^*(x) > 1 - \varepsilon \}
\]
be a slice of \( B_X \). Choose \( x \in S \cap S_X \) and find a sequence \( (y_n) \subset S_X \) such that  
\[
\|x \pm y_n\| \to 1.
\]
Then \( x \pm y_n \in S \) for large \( ns \), and then we get  
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\|x + y_n - (x - y_n)\| = 2\|y_n\| = 2.
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Similarly \( \omega US \Rightarrow D2P \) and \( US \Rightarrow SD2P \).
Sketch of proof.

Let $\varepsilon > 0$, $x^* \in S_{X^*}$, and

$S = S(x^*, \varepsilon) = \{ x \in B_X : x^*(x) > 1 - \varepsilon \}$ be a slice of $B_X$. Choose $x \in S \cap S_X$ and find a sequence $(y_n) \subset S_X$ such that $\|x \pm y_n\| \to 1$. Then $x \pm y_n \in S$ for large $n$s, and then we get

$$\|x + y_n - (x - y_n)\| = 2\|y_n\| = 2.$$

Similarly $\omega US \Rightarrow D2P$ and $US \Rightarrow SD2P$. 
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A characterization of \textit{LUS} spaces

\begin{theorem}
For a Banach space \( X \). TFAE.
\begin{enumerate}
  \item \( X \) is LUS.
  \item For every \( \varepsilon > 0 \) and every 1-dimensional subspace \( F \subset X \) there exists \( y \in S_X \) such that
    \[
    (1 - \varepsilon) \max\{\|x\|, |\lambda|\} \leq \|x + \lambda y\| \leq (1 + \varepsilon) \max\{\|x\|, |\lambda|\}
    \]
    for all \( x \in F \) and \( \lambda \in \mathbb{R} \).
\end{enumerate}
\end{theorem}

\begin{corollary}
LUS spaces contain almost isometric copies of \( \ell^2_\infty \).
\end{corollary}
A characterization of *LUS* spaces

**Theorem**

For a Banach space $X$. TFAE.

i) $X$ is *LUS*.

ii) For every $\varepsilon > 0$ and every 1-dimensional subspace $F \subset X$ there exists $y \in S_X$ such that

$$(1 - \varepsilon) \max\{\|x\|, |\lambda|\} \leq \|x + \lambda y\| \leq (1 + \varepsilon) \max\{\|x\|, |\lambda|\}$$

for all $x \in F$ and $\lambda \in \mathbb{R}$.

**Corollary**

*LUS* spaces contain almost isometric copies of $\ell^2_\infty$. 
A characterization of \( LUS \) spaces

**Theorem**

For a Banach space \( X \). TFAE.

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for all \( x \in F \) and \( \lambda \in \mathbb{R} \).

**Corollary**

\( LUS \) spaces contain almost isometric copies of \( \ell_2^2 \).
A characterization of US spaces

**Theorem**

*For a Banach space $X$. TFAE.*

i) $X$ is US.

ii) For every $\varepsilon > 0$, and finite dimensional subspace $F$ of $X$ there exist sequences $\varepsilon_n \downarrow 0$ and $(y_n)_{n=1}^{\infty}$ in $S_X$ such that

$$(1 - \varepsilon_n) \max\{\|f\|, |\lambda|\} \leq \|f + \lambda y_n\| \leq (1 + \varepsilon_n) \max\{\|f\|, |\lambda|\},$$

for every $f \in F_n = \text{span}\{F, (y_i)_{i=1}^{n-1}\}$ and $\lambda \in \mathbb{R}$. Moreover, $Y = \overline{\text{span}}(y_n)$ is $\varepsilon$-isometric to $c_0$. 
A characterization of \textit{US} spaces

\textbf{Theorem}

For a Banach space $X$. TFAE.

\begin{enumerate}[i)]
\item $X$ is US.
\item For every $\varepsilon > 0$, and finite dimensional subspace $F$ of $X$ there exist sequences $\varepsilon_n \downarrow 0$ and $(y_n)_{n=1}^{\infty}$ in $S_X$ such that
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(1 - \varepsilon_n) \max\{\|f\|, |\lambda|\} \leq \|f + \lambda y_n\| \leq (1 + \varepsilon_n) \max\{\|f\|, |\lambda|\},
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\end{enumerate}
Properties of \textit{US} spaces

Corollary

\textit{US} spaces contain almost isometric copies of \( c_0 \). (actually asymptotically isometric copies).

Corollary

The Cesaro function spaces \( C_p, 1 \leq p < \infty \) are not \textit{US}.

Corollary

\textit{If X is US, then} \( 0 \in \overline{\text{ext}^\omega \cdot B_{X^*}} \).
Properties of \textit{US} spaces

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Corollary

*If* $X$ *is US, then* $0 \in \overline{\text{ext}}^{\omega^*} B_{X^*}$. 
Proposition

If $X$ is US, then it is $\omega US$.

Sketch of proof.

Choose $(x_i)_{i=1}^N \subset S_F$. Then $\| (x_i, \pm e_n) \| = 1$ in $F \oplus_\infty c_0$. Define $S : F \oplus_\infty c_0 \to \text{span } (F, Y)$ by $S(f, x) = f + T(x)$ where $T : c_0 \to Y$ is the $\varepsilon$-isometry. Then $\| S(x_i, \pm e_n) \| = \| x_i \pm T(e_n) \| \to 1$, and $T(e_n) \to 0$ weakly in $X$ as $e_n \to 0$ weakly in $c_0$.  □
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\textbf{US} \implies \omega \textbf{US}

\textbf{Proposition}

\textit{If }X\text{ is US, then it is }\omega \text{US.}

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Choose \((x_i)_{i=1}^N \subset S_F\). Then \(\|(x_i, \pm e_n)\| = 1\) in \(F \oplus \infty c_0\). Define \(S : F \oplus \infty c_0 \rightarrow \text{span}(F, Y)\) by \(S(f, x) = f + T(x)\) where \(T : c_0 \rightarrow Y\) is the \(\varepsilon\)-isometry. Then \(\|S(x_i, \pm e_n)\| = \|x_i \pm T(e_n)\| \rightarrow 1\), and \(T(e_n) \rightarrow 0\) weakly in \(X\) as \(e_n \rightarrow 0\) weakly in \(c_0\). \qed
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Proposition

If $X$ is US, then it is $\omega US$.

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Choose $(x_i)_{i=1}^N \subset S_F$. Then $\|(x_i, \pm e_n)\| = 1$ in $F \oplus \infty c_0$. Define $S : F \oplus \infty c_0 \to \text{span} (F, Y)$ by $S(f, x) = f + T(x)$ where $T : c_0 \to Y$ is the $\epsilon$-isometry. Then $\|S(x_i, \pm e_n)\| = \|x_i \pm T(e_n)\| \to 1$, and $T(e_n) \to 0$ weakly in $X$ as $e_n \to 0$ weakly in $c_0$. \qed
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The intersection property - \textit{IP}

\textbf{Definition (Behrends and Harmand)}

\(X\) has the intersection property (\textit{IP}) if for all \(\varepsilon > 0\) there exist \((x_i)_{i=1}^{N} \subset B_X^o\) such that \(\|y\| \leq \varepsilon\) if \(\|x_i \pm y\| \leq 1\) for every \(i = 1, \ldots, N\), i.e. the intersection of the balls \(B(\pm x_i, 1)\) is contained in \(B(0, \varepsilon)\).

If \(X\) fails the \textit{IP} then there exists \(\varepsilon > 0\) such that for all \((x_i)_{i=1}^{N} \subset B_X^o\) there is \(y\) with \(\|y\| > \varepsilon\) and \(\|x_i \pm y\| \leq 1\) for every \(i = 1, \ldots, N\).
Definition (Behrends and Harmand)

$X$ has the intersection property (IP) if for all $\varepsilon > 0$ there exist $(x_i)_{i=1}^N \subset B_X^\circ$ such that $\|y\| \leq \varepsilon$ if $\|x_i \pm y\| \leq 1$ for every $i = 1, \ldots, N$, i.e. the intersection of the balls $B(\pm x_i, 1)$ is contained in $B(0, \varepsilon)$.

If $X$ fails the IP then there exists $\varepsilon > 0$ such that for all $(x_i)_{i=1}^N \subset B_X^\circ$ there is $y$ with $\|y\| > \varepsilon$ and $\|x_i \pm y\| \leq 1$ for every $i = 1, \ldots, N$. 
The intersection property - \textit{IP}

**Definition (Behrends and Harmand)**

$X$ has the intersection property (\textit{IP}) if for all $\varepsilon > 0$ there exist $(x_i)_{i=1}^N \subset B_X^\circ$ such that $\|y\| \leq \varepsilon$ if $\|x_i \pm y\| \leq 1$ for every $i = 1, \ldots, N$, i.e. the intersection of the balls $B(\pm x_i, 1)$ is contained in $B(0, \varepsilon)$.

If $X$ fails the \textit{IP} then there exists $\varepsilon > 0$ such that for all $(x_i)_{i=1}^N \subset B_X^\circ$ there is $y$ with $\|y\| > \varepsilon$ and $\|x_i \pm y\| \leq 1$ for every $i = 1, \ldots, N$. 

Proposition

\[ US \Rightarrow \text{fails the IP. But the converse is not true.} \]

Proof.

Characterization of \( US \). The \( G \)-space:

\[ X = \{ f \in C[0,1] : f(0) = 2f(1) \} \]

fails \( IP \) but is not \( US \) (not even \( LUS \)).
Proposition

**US** ⇒ *fails the IP*. But the converse is not true.

Proof.

Characterization of **US**. The *G*-space:

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$US \Rightarrow$ fails the $IP$. But the converse is not true.

Proof.

Characterization of $US$. The $G$-space:

$$X = \{ f \in C[0, 1] : f(0) = 2f(1) \}.$$  

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Can dual spaces be US?

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Can every space containing $c_0$ be renormed to be US?

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Does $LUS \Rightarrow \omega US$?
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