Uniformly square Banach spaces (joint work with J. Langemets and V. Lima)

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Examples Connection to diameter 2 spaces Characterizations *US* spaces fail the *IP* Open problems

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- X will denote a Banach space and X^* its dual.
- B_X the unit ball, B_X^o the open unit ball of X.

• S_X the unit sphere of X.

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Basic definitions

Definition (Schäffer, 1976)

A point $x \in S_X$ is called uniformly non-square if there exists $\delta > 0$ such that

 $\max \|x \pm y\| \ge 1 + \delta \text{ for all } y \in S_X.$

• If $x \in S_X$, then

 $2 \le ||x + y|| + ||x - y|| \le 2 \max ||x \pm y||$ for all $y \in X$

• So $x \in S_X$ is NOT uniformly non-square $\downarrow \downarrow$ there is $(y_n) \subset S_X$ with $||x \pm y_n|| \to 1$.

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X is

- a) locally uniformly square (*LUS*) if for every $x \in S_X$ there is a sequence $(y_n) \subset S_X$ such that $||x \pm y_n|| \to 1$.
- b) weakly uniformly square (ωUS) if X is LUS and $y_n \rightarrow 0$ weakly.
- c) uniformly square (US) if for every $(x_i)_{i=1}^N \subset S_X$ there is a sequence $(y_n) \subset S_X$ such that $||x_i \pm y_n|| \to 1$ for every $i = 1, \ldots, N$.
- $\bullet \ \mbox{Obviously b}) \Rightarrow a) \ \mbox{and c}) \Rightarrow a).$
- We will show: c) \Rightarrow b).

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- ... and ωUS as $e_n \rightarrow 0$ weakly.
- ... and US by the same idea as for the LUS case.
- Actually: $c_0(X_i)$ is US for any sequence of spaces X_i .

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Examples The space of convergent sequences - c

- Let $x = (1, 1, ..., 1, ...) \in S_c$. Now, if $||x \pm y_n|| \to 1$, then
 - to ± 1 , then the maximum of that term of $x \pm y_n$ would be close to 2. So *c* is not *LUS*.
- Actually ||y_n|| → 0. Reason: x = (1, 1, ..., 1, ...) is a strong extreme point in S_c.

Fact

Examples The space of convergent sequences - c

• Let $x = (1, 1, \dots, 1, \dots) \in S_c$. Now, if $||x \pm y_n|| \to 1$, then

 $||y_n|| \neq 1$. Because: if the value of one term of y_n were close to ± 1 , then the maximum of that term of $x \pm y_n$ would be close to 2. So *c* is not *LUS*.

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Let x = (1,1,...,1,...) ∈ S_c. Now, if ||x ± y_n|| → 1, then ||y_n|| ≠ 1. Because: if the value of one term of y_n were close to ±1, then the maximum of that term of x ± y_n would be close to 2. So c is not LUS.

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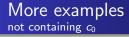
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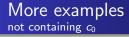


• $L_1[0,1]$ is ωUS , but not US.

Theorem (Kubiak, to appear)

The Cesaro function spaces C_p , $1 \le p < \infty$ are ωUS .

For $1 \le p < \infty$, $C_p = \{f \in L_p : \int_0^1 (1/x \int_0^x |f(t)| dt)^p < \infty\}$ with norm $||f|| = (\int_0^1 (1/x \int_0^x |f(t)| dt)^p)^{1/p}$.

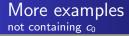


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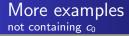


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Even more examples M-embedded spaces

• X is called M-embedded if $X^{***} = X^* \oplus_1 X^{\perp}$.

Theorem

Non-reflexive M-embedded spaces are US.

• In particular: $c_0(\Gamma)$, K(H), $K(\ell_p, \ell_q)$ where 1 are US.

• $c_0(\ell_1)$ is *US*, but not M-embedded.

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Definition

A Banach space X has the

- i) *local diameter 2 property (LD2P)* if every slice of B_X has diameter 2.
- ii) diameter 2 property (D2P) if every non-empty relatively weakly open subset of B_X has diameter 2.
- iii) strong diameter 2 property (SD2P) if every finite convex combination of slices of B_X has diameter 2.

Theorem (Becerra Guerrero, López-Pérez, Rueda Zoca)

 $LD2P \Rightarrow D2P$.

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Sketch of proof

Let $\varepsilon > 0$, $x^* \in S_{X^*}$, and $S = S(x^*, \varepsilon) = \{x \in B_X : x^*(x) > 1 - \varepsilon\}$ be a slice of B_X . Choose $x \in S \cap S_X$ and find a sequence $(y_n) \subset S_X$ such that $||x \pm y_n|| \to 1$. Then $x \pm y_n \in S$ for large *n*s, and then we get

$$||x + y_n - (x - y_n)|| = 2||y_n|| = 2.$$

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A characterization of LUS spaces

Theorem

For a Banach space X. TFAE.

i) X is LUS.

ii) For every $\varepsilon > 0$ and every 1-dimensional subspace $F \subset X$ there exists $y \in S_X$ such that

 $(1-\varepsilon)\max\{\|x\|, |\lambda|\} \le \|x+\lambda y\| \le (1+\varepsilon)\max\{\|x\|, |\lambda|\}$

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for all $x \in F$ and $\lambda \in \mathbb{R}$.

Corollary

LUS spaces contain almost isometric copies of ℓ_{∞}^2

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- ii) For every $\varepsilon > 0$ and every 1-dimensional subspace $F \subset X$ there exists $y \in S_X$ such that

 $(1-\varepsilon)\max\{\|x\|,|\lambda|\} \le \|x+\lambda y\| \le (1+\varepsilon)\max\{\|x\|,|\lambda|\}$

for all $x \in F$ and $\lambda \in \mathbb{R}$.

Corollary

LUS spaces contain almost isometric copies of ℓ_{∞}^2

A characterization of LUS spaces

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A characterization of US spaces

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For a Banach space X. TFAE.

i) X is US.

ii) For every ε > 0, and finite dimensional subspace F of X there exist sequences ε_n ↓ 0 and (y_n)_{n=1}[∞] in S_X such that

 $(1-\varepsilon_n)\max\{\|f\|,|\lambda|\} \le \|f+\lambda y_n\| \le (1+\varepsilon_n)\max\{\|f\|,|\lambda|\},$

for every $f \in F_n = span \{F, (y_i)_{i=1}^{n-1}\}$ and $\lambda \in \mathbb{R}$. Moreover, $Y = \overline{span}(y_n)$ is ε -isometric to c_0 .

A characterization of US spaces

Theorem

For a Banach space X. TFAE.

- i) X is US.
- ii) For every $\varepsilon > 0$, and finite dimensional subspace F of X there exist sequences $\varepsilon_n \downarrow 0$ and $(y_n)_{n=1}^{\infty}$ in S_X such that

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Properties of US spaces

Corollary

US spaces contain almost isometric copies of c_0 . (actually asymptotically isometric copies).

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The Cesaro function spaces C_p , $1 \le p < \infty$ are not US.

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If X is US, then it is ω US.

Sketch of proof.

Choose $(x_i)_{i=1}^N \subset S_F$. Then $||(x_i, \pm e_n)|| = 1$ in $F \oplus_\infty c_0$. Define $S : F \oplus_\infty c_0 \to \text{span}(F, Y)$ by S(f, x) = f + T(x) where $T : c_0 \to Y$ is the ε -isometry. Then $||S(x_i, \pm e_n)|| = ||x_i \pm T(e_n)|| \to 1$, and $T(e_n) \to 0$ weakly in X as $e_n \to 0$ weakly in c_0 .



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The intersection property - IP

Definition (Behrends and Harmand)

X has the intersection property (*IP*) if for all $\varepsilon > 0$ there exist $(x_i)_{i=1}^N \subset B_X^\circ$ such that $||y|| \le \varepsilon$ if $||x_i \pm y|| \le 1$ for every $i = 1, \ldots, N$, i.e. the intersection of the balls $B(\pm x_i, 1)$ is contained in $B(0, \varepsilon)$.

If X fails the *IP* then there exists $\varepsilon > 0$ such that for all $(x_i)_{i=1}^N \subset B_X^o$ there is y with $||y|| > \varepsilon$ and $||x_i \pm y|| \le 1$ for every $i = 1, \ldots, N$.

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Proposition

$US \Rightarrow$ fails the IP. But the converse is not true.

Proof.

Characterization of US. The G-space:

$$X = \{ f \in C[0,1] : f(0) = 2f(1) \}.$$

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Open problems

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Can dual spaces be US?

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Can every space containing c_0 be renormed to be US?

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