Asymptotic categories and their applications to projective operator modules

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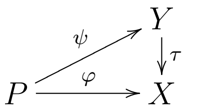
We shall discuss two approach to what to call a projective operator module, and a categorical framework, embracing these two kinds of projectivity, and many more.

The present talk is strongly influenced by two clusters of ideas and results. One is due to MacLane, and it is his theory of relative Abelian categories. The second source is the paper of David Blecher 'Standard dual etc.' of 1992, where, among other results, he introduced what he called 'projective operator spaces' and proved many things about their structure. The aim of this talk is to bind these things together. My point is that if we shall 'scratch' the results about projective operator spaces and modules, it will appear that they have, to the great extent, theoretic-categorical nature. Only at the very end of the relevant proofs the functional-analytic properties of our objects come to the forefront: we just show that the required theoretic-categorical assumptions are indeed fulfilled in our concrete context.

Let A be an algebra and simultaneously an operator space, or, as I prefer to say, quantum space. We say that A is a **quantum algebra**, if the bilinear operator of multiplication is completely bounded in the sense of the book of Effros and Ruan. Similarly, a left module over a quantum algebra A, which is simultaneously a quantum space, will be called **quantum A-module**, if the relevant bilinear operator of outer multiplication is completely bounded. The category of quantum A-modules and their morphisms that are completely contractive as operators is often denoted by $\mathbf{QA\text{-}mod_1}$, but I shall write, for brevity, $\mathcal{A}$.

An operator $\varphi : E \to F$ between quantum spaces is called **completely coisometric**, if all its amplifications are coisometric, that is they map the open unit ball of the domain space onto that of the range space. If we replace in this definition open balls by closed balls, we get the so-called **completely strictly coisometric** operator.

The first version the projectivity was introduced by Blecher. A quantum A-module P will be called **extremely projective**, if, for every completely coisometric morphism $\tau\colon Y \to X$, an arbitrary completely bounded morphism $\varphi\colon P \to X$ and every $\varepsilon > 0$ there exists a completely bounded morphism ψ (usually called a lifting of φ across τ), making the diagram



commutative and such that $\|\psi\|_{cb} < \|\varphi\|_{cb} + \varepsilon$. (Blecher, who considered completed operator spaces, used to say just 'projective'; we add 'extreme' because complete coisometries are exactly extreme epimorphisms, in categorical sense, in $\mathcal{A}$.

Another approach seems to be comparatively new. A quantum A-module P is called **metrically projective**, if, for every completely strictly coisometric morphism $\tau\colon Y \to X$ and an arbitrary completely bounded morphism $\varphi\colon P \to X$ there exists its lifting ψ across τ (see the previous diagram), such that $\|\psi\|_{cb} = \|\varphi\|_{cb}$.

Now we proceed to include these kinds of projectivity, at first metric, and then extreme, in some general-categorical scheme that will enable us to use the so-called free objects. Here I must recall a couple of things said two years ago; hope, it will not take much time.

Let $\mathcal{K}$ be an arbitrary category. A **rig** of $\mathcal{K}$ is a faithful covariant functor

$$\Box : \mathcal{K} \to \mathcal{L},$$

where $\mathcal{L}$ is another category. (We shall call $\mathcal{K}$ the main and $\mathcal{L}$ the auxiliary category). A category, equipped with a rig, is called **rigged category**. In such a context we call a morphism τ in $\mathcal{K}$ **admissible**, if $\Box(\tau)$ is a retraction in $\mathcal{L}$. Finally, an object P in $\mathcal{K}$ is called **projective**, if, for every admissible morphism $\tau : Y \to X$ and every morphism $\varphi : P \to X$, there exists a lifting, say ψ, of φ across τ.

Our main example is the rig $\odot$: $\mathbf{A\text{-}mod_1} \to \mathbf{Set}$ taking a module X to the cartesian product $\times_{n=1}^{\infty} \bigcirc_{M_n(X)}$ of closed unit balls of amplifications $M_n(X)$ of X and acting on morphisms in an obvious way. It is easy to verify that in this particular rigged category

admissible morphisms are exactly completely strictly coisomorphic A-module morphisms, and

projective objects are exactly metrically projective quantum modules.

Here we can present the first application (by no means deep one). Returning to the general-categorical context, one can easily show that the categorical coproduct of a family of projective objects in a rigged category is also projective. But it is well known, modulo the terminology, that the category $\mathcal{A}$ has coproducts, the so-called $\oplus_1$-sums (we shall say 'quantum l_1-sums'), introduced by Blecher. So we immediately have:

Proposition. *The $\oplus_1$-sum of a family of metrically projective quantum A-mudules is itself metrically projective.*

But what about the other kind of the projectivity, the extreme projectivity? The 'asymptotic' nature of the 'extreme' version, with the indispensable ε in its definition, requires a kind of elaboration of our scheme.

The key suggested notion is that of the **asymptotic category**. Let $\mathcal{K}$ be our main category. An **asymptotic structure** on $\mathcal{K}$ is the family, say $\{\mathbb{J}_\nu; \nu \in \Lambda\}$, of natural transformations of $\mathbf{1}_{\mathcal{K}}$, the identity functor on $\mathcal{K}$, satisfying two additional conditions. Let me recall what does this basic categorical notion mean. Namely, for every $\nu \in \Lambda$ and for every object X in $\mathcal{K}$ a morphism $\mathbb{J}_\nu^X : X \to X$ is given, such (and this is the sense of the word 'natural') that for every morphism $\varphi : X \to Y$ the diagram

$$\begin{array}{ccc} X & \xrightarrow{\varphi} & Y \\ {\scriptstyle \mathbb{J}_\nu^X}\downarrow & & \downarrow{\scriptstyle \mathbb{J}_\nu^Y} \\ X & \xrightarrow{\varphi} & Y \end{array}$$

is commutative. Two additional conditions are as follows:

(i) for every $\nu \in \Lambda$ and X, the morphism $\mathbb{J}_\nu^X$ is simultaneously mono- and epimorphism.

(ii) 'factorization': for every $\nu \in \Lambda$, there are $\lambda, \mu \in \Lambda$ such that $\mathbb{J}_\nu = \mathbb{J}_\lambda \mathbb{J}_\mu$.

An **asymptotic category** is a triple $(\mathcal{K}, \Box, \mathbb{J}_\nu)$, consisting of a category, a rig of the latter and an asymptotic structure on this category.

Speaking informally, the role of morphisms $\mathbb{J}_\nu^X$ is to form a set, 'approximating', in a sense, the identity morphism $\mathbf{1}_X$.

There is a lot of examples of asymptotic categories, but I shall present only one, connected with this talk. Taking our principal rigged category $\odot : \mathcal{A} \to \mathbf{Set}$, we set $\Lambda := (0,1)$ and, for $t \in (0,1)$, $\mathbb{J}_t^X := t\mathbf{1}_X$. It is easy to see that the triple $(\mathcal{A}, \odot, \mathbb{J}_t)$ is an asymptotic category.

Now return to the abstract asymptotic category $(\mathcal{K}, \Box, \mathbb{J}_\nu)$. We say that a morphism $\tau : Y \to X$ in $\mathcal{K}$ is **asymptotically admissible**, if, for every $\nu \in \Lambda$, there exists a morphism $\rho_\nu : \Box(X) \to \Box(Y)$ in $\mathcal{L}$ such that we have $\Box(\tau)\rho_\nu\Box(\mathbb{J}_\nu^X) = \Box(\mathbb{J}_\nu^X)$. It is easy to show that in our main example *asymptotically admissible morphisms are exactly complete coisometries.*

Finally, an object P in $\mathcal{K}$ is called **asymptotically projective** if, for every asymptotically admissible morphism τ, every morphism $\varphi : P \to X$ that admits a factorization $\varphi = \widetilde{\varphi}\mathbb{J}_\nu^X$, where $\widetilde{\varphi}$ is some other morphism, has a lifting across τ.

In our main example asymptotically projective objects turn out to be just extremely projective quantum A-modules.

As it was said, one benefit of the including of the discussed versions of the projectivity in the indicated categorical framework is that we can use the virtues of the so-called freedom. Let $\Box : \mathcal{K} \to \mathcal{L}$ be a rig, and M is an object of the auxiliary category $\mathcal{L}$. We call an object $F(M)$ in the main category $\mathcal{K}$ a **free** (or $\Box$-free if we need to specify the rig) **object with the base** M, if, for every $X \in \mathcal{K}$, there exists a bijection

$$\mathcal{I}_X : \mathbf{h}_{\mathcal{L}}(M, \Box X) \to \mathbf{h}_{\mathcal{K}}(F(M), X)$$

between the respective sets of morphisms, natural in the second argument X. ('Natural' means that some diagrams that appear when a morphism between two objects is given, are commutative). This is a slight generalization of the notion of a free object in a relative Abelian category, introduced by MacLane. We call a rigged category **freedom-loving**, if every object in $\mathcal{L}$ is a base of a free object in $\mathcal{K}$.

Proposition. *Let $(\mathcal{K}, \Box)$ is a freedom-loving rigged category. Then an object in the main category is projective if, and only if it is a retract of a free object. If, in addition, we have an asymptotic category $(\mathcal{K}, \Box, \mathbb{J}_\nu)$, then an object P in the main category is asymptotically projective if, and only if it is an asymptotic retract of a free object.* (The latter means that there exists a morphism $\sigma : F \to P$, where F is free, such that for every $\nu \in \Lambda$ there exist a morphism $\rho_\nu : P \to F$ with $\sigma\rho_\nu = \mathbb{J}_\nu^P$.)

What does it give to our concrete asymptotic category $(\mathcal{A}, \odot, \mathbb{J}_t)$? Blecher in his mentioned paper distinguished quantum spaces $T_n; n = 1, 2, \dots$ which are quantum dual to the standard quantum spaces $\mathcal{B}(\mathbb{C}^n)$. He observed some important property of these spaces that, after translation to categorical language and extending from just spaces to modules, sounds as follows:

Proposition. *Consider, for a natural n, the rig $\odot_n$: $\mathbf{A}$-$\mathbf{mod_1} \to \mathbf{Set}$, taking a module X to the closed unit ball $\bigcirc_{M_n(X)}$ of the n-th amplification $M_n(X)$ of X and acting on morphisms in an obvious way. Then $A \otimes_\wedge T_n$, where $\otimes_\wedge$ is the symbol of the operator-projective tensor product of quantum spaces, is the $\odot_n$-free module with the one-point base.*

After this, a purely categorical argument (I omit details) leads us to the following description of free objects in our principal example of a rigged category. Introduce the quantum space $T_\infty := T_1 \oplus_1 T_2 \cdots \oplus_1 T_n \cdots$.

Theorem. *The rigged category $\odot$: $\mathbf{A}$-$\mathbf{mod_1}$ $\to$ $\mathbf{Set}$ is freedom-loving. Moreover, for an arbitrary set M, the free quantum space with a base M is $\oplus_1\{(A \otimes_\wedge T_\infty)_t; t \in M\}$, that is the $\oplus_1$-sum of the family of copies of the module $A \otimes_\wedge T_\infty$, indexed by points of M.*

This, taken together with the characterization of projective and asymptotically projective objects in terms of free objects, formulated before, gives the description of metrically and extremely quantum modules. For brevity, the modules of the form $A \otimes_\wedge T_n$ for some n will be referred as **bricks**.

Corollary. *Let P is a quantum A-module. Then P is metrically (respectively, extremely) projective if, and only if for some quantum A-module Q which is the $\oplus_1$-sum of some family of bricks, there is a completely coisometric morphism $\sigma : F \to P$ that has a right inverse morphism ρ with $\|\rho\|_{cb} = 1$ (respectively, for every $\varepsilon > 0$ has a right inverse morphism ρ with $\|\rho\|_{cb} < 1 + \varepsilon$.)*

The part of the assertion, concerning the extreme projectivity, is actually due to Blecher, who considered complete quantum spaces.

THANK YOU