# Maximal left ideals of operators acting on a Banach space

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Göteborg, 29$^{th}$ July 2013

Joint work with Garth Dales and Niels Laustsen (both Lancaster), Tomasz Kochanek (University of Silesia, Poland) and Piotr Koszmider (IMPAN, Warsaw)

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*Let us specialise to $\mathscr{B}(E)$ then.*

**Question I.** Is this conjecture true for $\mathscr{A} = \mathscr{B}(E)$, the Banach algebra of all bounded, linear operators acting on a Banach space $E$?

# A partial answer to Question 1

**Theorem** (DKKKL). *Let $E$ be a separable Banach space (or just with $|E| = \mathfrak{c}$) with a countable, unconditional Schauder decomposition.*

*Then $\mathscr{B}(E)$ contains $2^{\mathfrak{c}}$ maximal left ideals, but only $\mathfrak{c}$ finitely-generated, maximal left ideals, where $\mathfrak{c} = 2^{\aleph_0}$.*

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**Observation.** *Let $E$ be a Banach space. For each $x \in E \setminus \{0\}$,*

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**Spoiler alert:** Yes, surprisingly it can happen! (To be revealed later what $E$ is.)

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*Then each finitely-generated, maximal left ideal of $\mathscr{B}(E)$ is fixed.*

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# A negative answer to Question II: Argyros–Haydon's Banach space

**Theorem** (Argyros–Haydon 2011). There is a Banach space $X_{\mathsf{AH}}$ which has the following three remarkable properties:

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Note that for $E = X_{\mathrm{AH}}$ everything goes well since it is an HI space hence condition (v) from the previous theorem applies.

# A negative answer to Question II: the example

Let $E = X_{\mathsf{AH}} \oplus \ell_\infty$.

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# A negative answer to Question II: the example continued

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More precisely, $\mathscr{K}_1$ is generated as a left ideal by the (single!) operator

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# How about Question 1?

Recall: $E = X_{\mathsf{AH}} \oplus \ell_\infty$.

**Theorem** (DKKKL). *The ideal* $\mathscr{K}_1$ *is the unique non-fixed, finitely-generated, maximal left ideal of* $\mathscr{B}(E)$.

# How about Question I?

Recall: $E = X_{\mathrm{AH}} \oplus \ell_\infty$.

**Theorem** (DKKKL). *The ideal $\mathscr{K}_1$ is the unique non-fixed, finitely-generated, maximal left ideal of $\mathscr{B}(E)$. Hence*

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*In particular, the answer to Question I is positive for $E$.*

# Open problems

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