Norms of idempotent Schur multipliers

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Outline

Schur multipliers

- The Schur product of $A, B \in M_{m\times n}$ is $A \bullet B = [a_{ij}b_{ij}]$ (a.k.a. the Hadamard product)
- The Schur multiplier corresponding to B is the linear map

$$S_B \colon M_{m\times n} \to M_{m\times n}, \quad S_B(A) = A \bullet B.$$

- We also consider "$m = n = \infty$": change $M_{m\times n}$ to $\mathcal{B}(\ell^2)$ and take infinite matrices B that give *bounded* maps S_B.
- These form a commutative semisimple Banach algebra.

Idempotent Schur multipliers

- In this talk B will be a matrix of 0s and 1s.
- Then $B \bullet B = B \implies S_B \circ S_B = S_B$, so S_B is idempotent

Motivating question

What are the possible values of $\|S_B : M_{m\times n} \to M_{m\times n}\|$?

- Trivially, 0 and 1 are possible values, but nothing in between:

$$\|S_B\| = \|S_B^2\| \leq \|S_B\|^2 \implies \|S_B\| \in \{0\} \cup [1, \infty).$$

- We can have $\|S_B\| > 1$:

Example

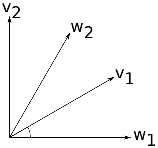
- $B = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ has $\|S_B\| \geq \|U \bullet B\| = \sqrt{\frac{4}{3}}$ for $U = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix}$.
- $C = B^{\otimes k}$ has $\|S_C\| = \|S_B\|^k \to \infty$ as $k \to \infty$

Showing that $\|S_{\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]}\| = \sqrt{\frac{4}{3}}$

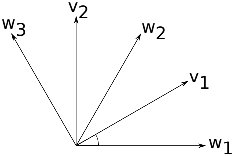
Theorem (Grothendieck)

$$\|S_B\| \le 1 \iff \exists\, v_i, w_j \in \mathrm{ball}(\ell^2)\colon B = [\langle v_i, w_j\rangle]$$

- $[\langle v_i, w_j\rangle] = \sqrt{\frac{3}{4}}\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]$



- So $\|S_{\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]}\| = \sqrt{\frac{4}{3}}\|S_{\sqrt{\frac{3}{4}}\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]}\| \le \sqrt{\frac{4}{3}}$.
- Have unitary U with $\|S_{\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]}(U)\| = \sqrt{\frac{4}{3}}$, so we have equality.
- In fact, $\|S_{\left[\begin{smallmatrix}1&1&0\\0&1&1\end{smallmatrix}\right]}\| = \sqrt{\frac{4}{3}}$ too:



“Diagonal + superdiagonal” idempotents

Let $B = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{bmatrix} \in M_{n\times n}$ and $C = \begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & 1 & 1 & \\ & & & & 1 & 1 \end{bmatrix} \in M_{n\times(n+1)}$.

Theorem (L., 2012)

$$\|S_B\| = \|S_C\| = \frac{2}{n+1}\cot\frac{\pi}{2(n+1)}.$$

Question

Given a matrix Y, which submatrices X have $\|S_X\| = \|S_Y\|$?

Example [Davidson–Donsig 2007]

For n odd, $D = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ 1 & & & & 1 \end{bmatrix} \in M_{(n+1)\times(n+1)}$ has $\|S_D\| = \frac{2}{n+1}\cot\frac{\pi}{2(n+1)}$.

Note that B and C are both submatrices.

Livshits' two gaps theorem

Theorem (Livshits, 1995)

For any 0–1 *matrix B, we have* $\|S_B\| \in \{0,1\} \cup \left[\sqrt{\frac{4}{3}}, \infty\right)$.

We can say more: there are at least six gaps.

Bipartite graphs

$$\{\text{0–1 matrices in } M_{m\times n}\} \longleftrightarrow \{\text{(m,n) bipartite graphs}\}$$

$$\text{rows and columns} \longleftrightarrow \text{vertices}$$

$$\text{entries equal to 1} \longleftrightarrow \text{edges}$$

Examples

$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \longleftrightarrow G_B =$ 

$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \longleftrightarrow G_C =$ 

$B \oplus C = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \longleftrightarrow G_{B\oplus C} =$ 

Dictionary of operations

0–1 **matrix** B	**bipartite graph** G_B	**norm** $\|S_B\|$
shuffle rows or columns	bipartite graph isomorphism	equal
duplicate rows or columns	duplicate vertices and their edges	equal
submatrix	induced subgraph	decreases
direct sum	disjoint union	max

A proof of Livshits' theorem in this language

Theorem (Livshits, 1995)

For any 0–1 *matrix* B*, we have* $\|S_B\| \in \{0,1\} \cup \left[\sqrt{\frac{4}{3}}, \infty\right)$*.*

Proof.

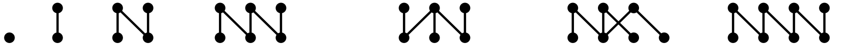
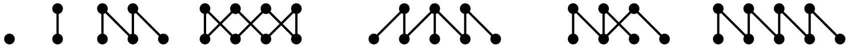
- Let $G = G_B$. WLOG: G is connected.
- If G is complete bipartite then $\|S_B\| \in \{0,1\}$.
- Otherwise, take vertices c and r so that:
  - c and r are in different parts of the bipartition;
  - (r, c) is not an edge of G; and
  - the distance from c to r is as small as possible
- A minimal path joining c to r starts with the subgraph 
- Minimality $\implies$ c and r are the ends of this path
- (c, r) not an edge of G $\implies$  is an induced subgraph of G

$$\implies \|S_B\| \geq \sqrt{\frac{4}{3}}.$$

□

Small idempotent Schur multipliers

Define η_k, E_k, F_k for $1 \le k \le 6$ as follows:

k	0	1	2	3	4	5	6
η_k	0	1	$\sqrt{\frac{4}{3}}$	$\frac{1+\sqrt{2}}{2}$	$\frac{1}{15}\sqrt{169+38\sqrt{19}}$	$\sqrt{\frac{3}{2}}$	$\frac{2}{5}\sqrt{5+2\sqrt{5}}$
E_k							
F_k							

Theorem (L., 2012)

If $G = G_B$ is a connected, duplicate-free bipartite graph and $1 \le k \le 6$, then the following are equivalent:

1. $\|S_B\| = \eta_k$
2. $\eta_{k-1} < \|S_B\| \le \eta_k$
3. $E_k \le G \le F_k$

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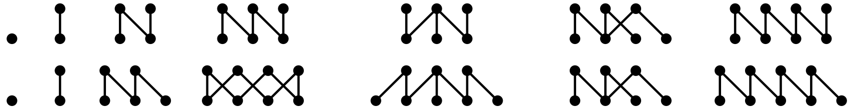
E_k

F_k



Corollary

If $G = G_B$ is any bipartite graph and $1 \le k \le 6$, then the following are equivalent:

1. $\|S_B\| = \eta_k$
2. $\eta_{k-1} < \|S_B\| \le \eta_k$
3. (i) *Some component H of $\mathrm{df}(G)$ has $E_k \le H \le F_k$, and*
   (ii) *Every component H of $\mathrm{df}(G)$ has $E_j \le H \le F_j$ for some $j \le k$.*

Six gaps

Corollary

If S_B is any idempotent Schur multiplier, then

$$\|S_B\| \in \{\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5\} \cup [\eta_6, \infty).$$

Using tools of Katavolos–Paulsen (2005), this generalises:

Theorem

The same is true if we replace S_B with any idempotent normal masa-bimodule map $S\colon B(H) \to B(H)$ where H is a separable Hilbert space.

Some natural questions

- $\mathcal{N} = \{\|S_B\| : B \in \{0,1\}^{m\times n},\ m,n \in \mathbb{N}\cup\{\infty\}\,\}$ contains left accumulation points, such as $4/\pi = \lim_{n\to\infty} \frac{2}{n+1}\cot(\frac{\pi}{2(n+1)})$.
  Is $4/\pi$ the smallest accumulation point in $\mathcal{N}$?
- Is $\mathcal{N}$ countable?
- Does $\mathcal{N}$ contain an open interval?
- Which graphs give Schur idempotents of the same norm?
- Find a combinatorial characterisation of the idempotent Schur multipliers on $B(\ell^2)$.

Thank you!