Norms of idempotent Schur multipliers

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Banach Algebras and Applications

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Outline

1. Schur multipliers
2. Some norm calculations
3. Gaps in the set of norms
Schur multipliers

- The Schur product of $A, B \in M_{m \times n}$ is $A \bullet B = [a_{ij}b_{ij}]$ (a.k.a. the Hadamard product)

- The Schur multiplier corresponding to $B$ is the linear map

\[ S_B : M_{m \times n} \to M_{m \times n}, \quad S_B(A) = A \bullet B. \]

- We also consider “$m = n = \infty$”: change $M_{m \times n}$ to $B(\ell^2)$ and take infinite matrices $B$ that give bounded maps $S_B$.

- These form a commutative semisimple Banach algebra.
Idempotent Schur multipliers

- In this talk $B$ will be a matrix of 0s and 1s.
- Then $B \bullet B = B \implies S_B \circ S_B = S_B$, so $S_B$ is idempotent.

Motivating question

What are the possible values of $\|S_B : M_{m \times n} \to M_{m \times n}\|$?

- Trivially, 0 and 1 are possible values, but nothing in between:
  \[
  \|S_B\| = \|S_B^2\| \leq \|S_B\|^2 \implies \|S_B\| \in \{0\} \cup [1, \infty).
  \]
- We can have $\|S_B\| > 1$:
  \[
  B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ has } \|S_B\| \geq \|U \bullet B\| = \sqrt{\frac{4}{3}} \text{ for } U = \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & 1 \\ -1 & \sqrt{2} \end{bmatrix}.
  \]
  \[
  C = B \otimes^k \text{ has } \|S_C\| = \|S_B\|^k \to \infty \text{ as } k \to \infty.
  \]
Showing that \( \| S_{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} } \| = \sqrt{\frac{4}{3}} \)

**Theorem (Grothendieck)**

\[ \| S_B \| \leq 1 \iff \exists \ v_i, w_j \in \text{ball}(\ell^2): B = [\langle v_i, w_j \rangle] \]

- \[ [\langle v_i, w_j \rangle] = \sqrt{\frac{3}{4}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \]
- So \( \| S_{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} } \| = \sqrt{\frac{4}{3}} \| S_{ \sqrt{\frac{3}{4}} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} } \| \leq \sqrt{\frac{4}{3}} \).
- Have unitary \( U \) with \( \| S_{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} } (U) \| = \sqrt{\frac{4}{3}} \), so we have equality.
- In fact, \( \| S_{ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} } \| = \sqrt{\frac{4}{3}} \) too:
“Diagonal + superdiagonal” idempotents

Let $B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \in M_{n \times n}$ and $C = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \in M_{n \times (n+1)}$.

Theorem (L., 2012)

$$\|S_B\| = \|S_C\| = \frac{2}{n+1} \cot \frac{\pi}{2(n+1)}.$$

Question

Given a matrix $Y$, which submatrices $X$ have $\|S_X\| = \|S_Y\|$?

Example [Davidson–Donsig 2007]

For $n$ odd, $D = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \in M_{(n+1) \times (n+1)}$ has $\|S_D\| = \frac{2}{n+1} \cot \frac{\pi}{2(n+1)}$.

Note that $B$ and $C$ are both submatrices.
Livshits’ two gaps theorem

Theorem (Livshits, 1995)

For any $0–1$ matrix $B$, we have $\|S_B\| \in \{0, 1\} \cup \left[\sqrt{\frac{4}{3}}, \infty\right)$.

We can say more: there are at least six gaps.
Bipartite graphs

\{0–1 matrices in \( M_{m \times n} \) \} \leftrightarrow \{ (m,n) bipartite graphs \}

rows and columns \leftrightarrow \text{vertices}

entries equal to 1 \leftrightarrow \text{edges}

**Examples**

\[ B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \leftrightarrow G_B = \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \end{array} \]

\[ C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \leftrightarrow G_C = \begin{array}{cc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{array} \]

\[ B \oplus C = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \leftrightarrow G_{B \oplus C} = \begin{array}{cc} \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \bullet \end{array} \]
# Dictionary of operations

<table>
<thead>
<tr>
<th>0–1 matrix $B$</th>
<th>bipartite graph $G_B$</th>
<th>norm $|S_B|$</th>
</tr>
</thead>
<tbody>
<tr>
<td>shuffle rows or columns</td>
<td>bipartite graph isomorphism</td>
<td>equal</td>
</tr>
<tr>
<td>duplicate rows or columns</td>
<td>duplicate vertices and their edges</td>
<td>equal</td>
</tr>
<tr>
<td>submatrix</td>
<td>induced subgraph</td>
<td>decreases</td>
</tr>
<tr>
<td>direct sum</td>
<td>disjoint union</td>
<td>max</td>
</tr>
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</table>
A proof of Livshits’ theorem in this language

**Theorem (Livshits, 1995)**

For any 0–1 matrix $B$, we have $\|S_B\| \in \{0, 1\} \cup \left[\sqrt{\frac{4}{3}}, \infty\right)$. 

**Proof.**

- Let $G = G_B$. WLOG: $G$ is connected.
- If $G$ is complete bipartite then $\|S_B\| \in \{0, 1\}$.
- Otherwise, take vertices $c$ and $r$ so that:
  - $c$ and $r$ are in different parts of the bipartition;
  - $(r, c)$ is not an edge of $G$; and
  - the distance from $c$ to $r$ is as small as possible
- A minimal path joining $c$ to $r$ starts with the subgraph $\mathcal{N}$
- Minimality $\implies c$ and $r$ are the ends of this path
- $(c, r)$ not an edge of $G$ $\implies \mathcal{N}$ is an induced subgraph of $G$
  $\implies \|S_B\| \geq \sqrt{\frac{4}{3}}$. 

Rupert Levene (Dublin)
Small idempotent Schur multipliers

Define $\eta_k$, $E_k$, $F_k$ for $1 \leq k \leq 6$ as follows:

<table>
<thead>
<tr>
<th>$k$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<tr>
<td>$\eta_k$</td>
<td>0</td>
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<td>$\frac{1}{15} \sqrt{169 + 38\sqrt{19}}$</td>
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Theorem (L., 2012)

If $G = G_B$ is a connected, duplicate-free bipartite graph and $1 \leq k \leq 6$, then the following are equivalent:

1. $\|S_B\| = \eta_k$
2. $\eta_{k-1} < \|S_B\| \leq \eta_k$
3. $E_k \leq G \leq F_k$
Small idempotent Schur multipliers

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<tr>
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</tbody>
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Corollary

If $G = G_B$ is any bipartite graph and $1 \leq k \leq 6$, then the following are equivalent:

1. $\| S_B \| = \eta_k$
2. $\eta_{k-1} < \| S_B \| \leq \eta_k$
3. (i) Some component $H$ of $df(G)$ has $E_k \leq H \leq F_k$, and
   (ii) Every component $H$ of $df(G)$ has $E_j \leq H \leq F_j$ for some $j \leq k$. 
Six gaps

**Corollary**

*If $S_B$ is any idempotent Schur multiplier, then*

\[ \|S_B\| \in \{\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5\} \cup [\eta_6, \infty). \]

**Using tools of Katavolos–Paulsen (2005), this generalises:**

**Theorem**

*The same is true if we replace $S_B$ with any idempotent normal masa-bimodule map $S: B(H) \to B(H)$ where $H$ is a separable Hilbert space.*
Some natural questions

\[ \mathcal{N} = \{ \|S_B\| : B \in \{0, 1\}^{m \times n}, \ m, n \in \mathbb{N} \cup \{\infty\} \} \] contains left accumulation points, such as \( \frac{4}{\pi} = \lim_{n \to \infty} \frac{2}{n+1} \cot\left(\frac{\pi}{2(n+1)}\right) \).

Is \( \frac{4}{\pi} \) the smallest accumulation point in \( \mathcal{N} \)?

- Is \( \mathcal{N} \) countable?
- Does \( \mathcal{N} \) contain an open interval?
- Which graphs give Schur idempotents of the same norm?
- Find a combinatorial characterisation of the idempotent Schur multipliers on \( B(\ell^2) \).
Thank you!