C^*-Segal algebras with order unit

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Some literature

- Jukka Kauppi and Martin Mathieu: *C^*-Segal algebras with order unit*, J. Math. Anal. Appl. **398** (2013), 785–797.
- Jorma Arhippainen and Jukka Kauppi: *On dense ideals of C^*-algebras and generalizations of the Gelfand–Naimark theorem*, Studia Math. **215** (2013), 71–98.
- Jussi Mattas: *Segal algebras, approximate identities and norm irregularity in $C_0(X, A)$*, Studia Math. **215** (2013), 99–112.

A^*-algebras

throughout A will be a Banach algebra with norm $\|\cdot\|$ and involution $*$

A is an A^*-algebra if it carries a (possibly different) C^*-norm $\|\cdot\|_C$ (so that $C = (A, \|\cdot\|_C)^\sim$ is a C^*-algebra).

Facts: let A be an A^*-algebra; then

- the involution on A is continuous;

  thus can WLOG assume that the involution is isometric and that $\|x\|_C \leq \|x\| \;\forall\, x \in A$.
- A is semisimple;
- A has a faithful $*$-representation on Hilbert space which is isometric w.r.t. $\|\cdot\|_C$.

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Examples:

- $L^1(G) \subseteq C^*(G)$ for any locally compact group G;
- $C_p \subseteq K(H)$, $p \geq 1$ the Schatten p-classes;
- $\ell^p \subseteq c_0$, $p \geq 1$.

C^*-Segal algebras

Definition

An A^*-algebra A is called a *self-adjoint C^*-Segal algebra* if it is an ideal in its surrounding C^*-algebra $C = (A, \|\cdot\|_C)^\sim$.

Remark (Kassem–Rowlands 1987)

The auxiliary norm of a C^*-Segal algebra is unique.

Proposition

*A Banach *-algebra A is a self-adjoint C^*-Segal algebra if and only if there is a (continuous) injective *-homomorphism from A into a C^*-algebra C such that the image of A is a dense ideal in C.*

Alexander, Barnes, Burnham, Kassem–Rowlands, Leinert, Tomiuk,. . .

C^*-Segal algebras

Definition

An A^*-algebra A is called a *self-adjoint C^*-Segal algebra* if it is an ideal in its surrounding C^*-algebra $C = (A, \|\cdot\|_C)^\sim$.

Question: How to locate A inside C?

C^*-Segal algebras

Example:
Let X be a locally compact Hausdorff space, and $C = C_0(X)$. Let $\omega\colon X \to \mathbb{R}$ be an upper semicontinuous function such that $\omega(t) \geq 1$ for every $t \in X$. Define

$$C_b^\omega(X) = \{f \in C(X) :\ \omega f \text{ is bounded on } X\}$$
$$C_0^\omega(X) = \{f \in C(X) :\ \omega f \text{ vanishes at infinity on } X\},$$

where $C(X)$ denotes all continuous complex-valued functions on X. Equipped with pointwise operations and the *weighted supremum norm*

$$\|f\|_\omega := \sup_{t\in X} \omega(t)|f(t)|,$$

$C_b^\omega(X)$ and $C_0^\omega(X)$ are self-adjoint C^*-Segal algebras.

C^*-Segal algebras with order unit

the key to understanding the algebras $C_b^\omega(X)$ and $C_0^\omega(X)$ is the fact that $\frac{1}{\omega}$ is an order unit in $C_b^\omega(X)$ whenever it is continuous on X;

endow the self-adjoint C^*-Segal algebra $A \subseteq C$ with the canonical order

$$x \leq y \quad \text{if} \quad y - x \in A_+ := A \cap C_+ \quad (x, y \in A_h),$$

where A_h denote the real vector space of self-adjoint elements of A.

An element $u \in A_+$ is called an *order unit* of A if each $x \in A_h$ satisfies $x \leq \gamma u$ for some constant $\gamma > 0$.

Such u *strictly positive*, i.e., $\varphi(u) > 0$ for every positive functional $\varphi \neq 0$ on A.

Order unit C^*-Segal algebras

Definition

By an *order unit C^*-Segal algebra* we mean a pair (A, u), where A is a self-adjoint C^*-Segal algebra and u is an order unit of A satisfying

$$\|a\| = \inf\{\gamma > 0 : \; -\gamma u \leq a \leq \gamma u\}$$

for all $a \in A_h$.

A characterisation

Theorem (Kauppi–Mathieu)

Let A be a C^*-Segal algebra in the C^*-algebra C, and let $u \in A_+$ be strictly positive. Put $v = u^{\frac{1}{2}} \in C_+$. The following conditions are equivalent:

(a) *(A, u) is an order unit C^*-Segal algebra;*

(b) *there exists a self-adjoint C-subbimodule D of $M(C)$ containing C and 1 such that $A = vDv$, $vC = Cv$ and $\|vdv\| = \|d\|_C$ for all $d \in D_h$.*

In this case, the surrounding C^*-algebra C is σ-unital (i.e., it contains a countable contractive approximate identity) and $E_A = vCv = uC = Cu$ and $M_C(A) = vM(C)v = uM(C) = M(C)u$.

The commutative case

Theorem (Arhipainnen–Kauppi)

*Let A in the above theorem be commutative. Then A is isometrically *-isomorphic to a closed self-adjoint subalgebra of $C_b^\omega(X)$ for a locally compact Hausdorff space X and a continuous real-valued function ω on X with $\omega(t) \geq 1$ for all $t \in X$.*
*In particular, up to an isometric *-isomorphism, $E_A = C_0^\omega(X)$, $M_C(A) = C_b^\omega(X)$ and $M(C) = C_b(X)$.*

Multiplier modules

let A be a (not necessarily self-adjoint) C^*-Segal algebra in the C^*-algebra C and let $M(C)$ denote the multiplier algebra of C;

let $M_C(A) = \{m \in M(C) \colon mC + Cm \subseteq A\}$ be the *multiplier module of A with respect to C*;

then $M_C(A)$ is a Banach subalgebra of $\mathscr{L}(C, A)$ and a Banach C-bimodule;

under the strict topology defined by the semi-norms

$$m \mapsto \|mx\| + \|xm\| \qquad (x \in C)$$

$M_C(A)$ is a complete locally convex algebra.

The approximate ideal

Lemma

Let A be a C^*-Segal algebra in the C^*-algebra C. Then

$$\|a\|_M := \sup_{\|b\|\leq 1} \{\|ab\|, \|ba\|\} \qquad (a, b \in A)$$

defines a norm $\|\cdot\|_M$ which is equivalent to $\|\cdot\|_C$ on A. Furthermore, $A_M = (A, \|\cdot\|_M)$ has a bounded approximate identity which is contractive under the norm on C.

The approximate ideal

let $\widetilde{A}_M$ denote the completion of A_M;

Proposition

Let A be a C^*-Segal algebra in the C^*-algebra C and let $(e_\alpha)_{\alpha\in\Omega}$ be a bounded approximate identity in A_M. Then

(i) $A\widetilde{A}_M = \widetilde{A}_M A$ *is a closed ideal in A;*

(ii) $A\widetilde{A}_M = \{a \in A : \ \|ae_\alpha - a\| \to 0 \text{ and } \|e_\alpha a - a\| \to 0\}$;

(iii) $A\widetilde{A}_M$ *has an approximate identity;*

(iv) *every closed ideal of A with an approximate identity is contained in $A\widetilde{A}_M$.*

The approximate ideal

Definition

We put $E_A := A\widetilde{A}_M$ and call it the *approximate ideal* of A.

Example: let $A = C_0^\omega(X)$ with ω as above; then

$C = C_0(X)$, $M(C) = C_b(X)$, $E_A = A$ and $M_C(A) = C_b^\omega(X)$.

Example (Mattas):

let $A = C_0(X, B)$ where B is a C^*-Segal algebra in C; then

A is a C^*-Segal algebra in $C_0(X, C)$ and $E_A = C_0(X, E_B)$.

The approximate ideal

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We put $E_A := A\widetilde{A}_M$ and call it the *approximate ideal* of A.

Some properties of E_A:

(i) A^2 is dense in E_A;

(ii) $AC = CA = E_A$;

(iii) E_A is a C^*-Segal algebra in C;

(iv) E_A is strictly dense in $M_C(A)$;

(v) if A is a self-adjoint C^*-Segal algebra then both E_A and $M_C(A)$ are self-adjoint C^*-Segal algebras.

$$E_A \subseteq A \subseteq C \subseteq M_C(A) \subseteq M(C).$$

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In this case, the surrounding C^*-algebra C is σ-unital (i.e., it contains a countable contractive approximate identity) and $E_A = vCv = uC = Cu$ and $M_C(A) = vM(C)v = uM(C) = M(C)u$.

An application

Let B be a C^*-algebra and let $u \in Z(M(B))_+$ be such that uB is faithful in B.

Put $A = uB$ and $C = \overline{A}^{\|\cdot\|_B}$. Then A is a self-adjoint C^*-Segal algebra in C under the norm $\|ux\|_u := \|x\|_B$ for $x \in B$.

It follows that $M_C(A)$ is an order unit C^*-Segal algebra containing A isometrically as a faithful ideal (an order unitisation of A).

Moreover, $E_A = AC = uBC = uC$ and hence uB has an approximate identity if and only if it is dense in B.