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**C*-Segal algebras with order unit**

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Gothenburg, 30 July 2013
Some literature


$A^*$-algebras

Throughout $A$ will be a Banach algebra with norm $\| \cdot \|$ and involution $^*$.

$A$ is an $A^*$-algebra if it carries a (possibly different) $C^*$-norm $\| \cdot \|_C$ (so that $C = (A, \| \cdot \|_C) \sim$ is a $C^*$-algebra).

**Facts:** let $A$ be an $A^*$-algebra; then

- the involution on $A$ is continuous;
  thus can WLOG assume that the involution is isometric and that $\| x \|_C \leq \| x \| \quad \forall x \in A$.
- $A$ is semisimple;
- $A$ has a faithful $^*$-representation on Hilbert space which is isometric w.r.t. $\| \cdot \|_C$. 

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$C^*$-Segal algebras with order unit
$A^*$-algebras

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$A$ is an $A^*$-algebra if it carries a (possibly different) $C^*$-norm $\| \cdot \|_C$ (so that $C = (A, \| \cdot \|_C)^\sim$ is a $C^*$-algebra).

Examples:

- $L^1(G) \subseteq C^*(G)$ for any locally compact group $G$;
- $C_p \subseteq K(H), \ p \geq 1$ the Schatten $p$-classes;
- $\ell^p \subseteq c_0, \ p \geq 1$. 
**C*-Segal algebras**

**Definition**

An $A^*$-algebra $A$ is called a *self-adjoint C*-Segal algebra* if it is an ideal in its surrounding $C^*$-algebra $C = (A, \| \cdot \| C)$. 

**Remark (Kassem–Rowlands 1987)**

The auxiliary norm of a $C^*$-Segal algebra is unique.

**Proposition**

A Banach $^*$-algebra $A$ is a self-adjoint $C^*$-Segal algebra if and only if there is a (continuous) injective $^*$-homomorphism from $A$ into a $C^*$-algebra $C$ such that the image of $A$ is a dense ideal in $C$.

Alexander, Barnes, Burnham, Kassem–Rowlands, Leinert, Tomiuk, …
Definition
An $A^*$-algebra $A$ is called a *self-adjoint C*-Segal algebra* if it is an ideal in its surrounding C*-algebra $C = (A, \| \cdot \|_C)$. 

**Question:** How to locate $A$ inside $C$?

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*C*-Segal algebras with order unit*
**C*-Segal algebras**

**Example:**
Let $X$ be a locally compact Hausdorff space, and $C = C_0(X)$. Let $\omega : X \to \mathbb{R}$ be an upper semicontinuous function such that $\omega(t) \geq 1$ for every $t \in X$. Define

\[
C_b^\omega(X) = \{ f \in C(X) : \omega f \text{ is bounded on } X \}
\]
\[
C_0^\omega(X) = \{ f \in C(X) : \omega f \text{ vanishes at infinity on } X \},
\]

where $C(X)$ denotes all continuous complex-valued functions on $X$. Equipped with pointwise operations and the *weighted supremum norm*

\[
\| f \|_\omega := \sup_{t \in X} \omega(t) |f(t)|,
\]

$C_b^\omega(X)$ and $C_0^\omega(X)$ are self-adjoint C*-Segal algebras.
The key to understanding the algebras $C^\omega_b(X)$ and $C^\omega_0(X)$ is the fact that $\frac{1}{\omega}$ is an order unit in $C^\omega_b(X)$ whenever it is continuous on $X$;

endow the self-adjoint $C^*$-Segal algebra $A \subseteq C$ with the canonical order

$$x \leq y \text{ if } y - x \in A_+ := A \cap C_+ \quad (x, y \in A_h),$$

where $A_h$ denote the real vector space of self-adjoint elements of $A$. An element $u \in A_+$ is called an order unit of $A$ if each $x \in A_h$ satisfies $x \leq \gamma u$ for some constant $\gamma > 0$.

Such $u$ strictly positive, i.e., $\varphi(u) > 0$ for every positive functional $\varphi \neq 0$ on $A$. 

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Order unit $C^*$-Segal algebras

**Definition**

By an *order unit $C^*$-Segal algebra* we mean a pair $(A, u)$, where $A$ is a self-adjoint $C^*$-Segal algebra and $u$ is an order unit of $A$ satisfying

$$
\|a\| = \inf\{\gamma > 0 : -\gamma u \leq a \leq \gamma u\}
$$

for all $a \in A_h$. 
A characterisation

**Theorem (Kauppi–Mathieu)**

Let $A$ be a $C^*$-Segal algebra in the $C^*$-algebra $C$, and let $u \in A_+$ be strictly positive. Put $v = u^{1/2} \in C_+$. The following conditions are equivalent:

(a) $(A, u)$ is an order unit $C^*$-Segal algebra;

(b) there exists a self-adjoint $C$-subbimodule $D$ of $M(C)$ containing $C$ and $1$ such that $A = vDv$, $vC = Cv$ and $\|vdv\| = \|d\|_C$ for all $d \in D_h$.

In this case, the surrounding $C^*$-algebra $C$ is $\sigma$-unital (i.e., it contains a countable contractive approximate identity) and $E_A = vCv = uC = Cu$ and $M_C(A) = vM(C)v = uM(C) = M(C)u$. 
The commutative case

**Theorem (Arhipainen–Kauppi)**

Let $A$ in the above theorem be commutative. Then $A$ is isometrically *-isomorphic to a closed self-adjoint subalgebra of $C^\omega_b(X)$ for a locally compact Hausdorff space $X$ and a continuous real-valued function $\omega$ on $X$ with $\omega(t) \geq 1$ for all $t \in X$.

In particular, up to an isometric *-isomorphism, $E_A = C^\omega_0(X)$, $M_C(A) = C^\omega_b(X)$ and $M(C) = C_b(X)$. 
Multiplier modules

let \( A \) be a (not necessarily self-adjoint) \( C^* \)-Segal algebra in the \( C^* \)-algebra \( C \) and let \( M(C) \) denote the multiplier algebra of \( C \);

let \( M_C(A) = \{ m \in M(C) : mC + Cm \subseteq A \} \) be the \textit{multiplier module of \( A \) with respect to \( C \)};

then \( M_C(A) \) is a Banach subalgebra of \( \mathcal{L}(C, A) \) and a Banach \( C \)-bimodule;

under the \textit{strict topology} defined by the semi-norms

\[
m \mapsto \| mx \| + \| xm \| \quad (x \in C)
\]

\( M_C(A) \) is a complete locally convex algebra.
The approximate ideal

**Lemma**

Let $A$ be a $C^*$-Segal algebra in the $C^*$-algebra $C$. Then

$$
\|a\|_M := \sup_{\|b\| \leq 1} \{ \|ab\|, \|ba\| \} \quad (a, b \in A)
$$

defines a norm $\| \cdot \|_M$ which is equivalent to $\| \cdot \|_C$ on $A$. Furthermore, $A_M = (A, \| \cdot \|_M)$ has a bounded approximate identity which is contractive under the norm on $C$. 
The approximate ideal

Let \( \tilde{A}_M \) denote the completion of \( A_M \);

**Proposition**

Let \( A \) be a \( C^* \)-Segal algebra in the \( C^* \)-algebra \( C \) and let \( (e_\alpha)_{\alpha \in \Omega} \) be a bounded approximate identity in \( A_M \). Then

(i) \( \tilde{A}_M A = \tilde{A}_M A \) is a closed ideal in \( A \);

(ii) \( \tilde{A}_M \) is the set \( \{ a \in A : \| ae_\alpha - a \| \to 0 \text{ and } \| e_\alpha a - a \| \to 0 \} \);

(iii) \( \tilde{A}_M \) has an approximate identity;

(iv) every closed ideal of \( A \) with an approximate identity is contained in \( \tilde{A}_M \).
The approximate ideal

**Definition**

We put $E_A := \widetilde{A\tilde{A}}_M$ and call it the *approximate ideal* of $A$.

**Example:** let $A = C_0^\omega(X)$ with $\omega$ as above; then $C = C_0(X)$, $M(C) = C_b(X)$, $E_A = A$ and $M_C(A) = C_b^\omega(X)$.

**Example (Mattas):**

let $A = C_0(X, B)$ where $B$ is a $C^*$-Segal algebra in $C$; then $A$ is a $C^*$-Segal algebra in $C_0(X, C)$ and $E_A = C_0(X, E_B)$. 
The approximate ideal

Definition

We put $E_A := \tilde{A}_M$ and call it the *approximate ideal* of $A$.

Some properties of $E_A$:

(i) $A^2$ is dense in $E_A$;
(ii) $AC = CA = E_A$;
(iii) $E_A$ is a $C^*$-Segal algebra in $C$;
(iv) $E_A$ is strictly dense in $M_C(A)$;
(v) if $A$ is a self-adjoint $C^*$-Segal algebra then both $E_A$ and $M_C(A)$ are self-adjoint $C^*$-Segal algebras.

$E_A \subseteq A \subseteq C \subseteq M_C(A) \subseteq M(C)$. 
A characterisation

**Theorem (Kauppi–Mathieu)**

Let $A$ be a $\mathcal{C}^*$-Segal algebra in the $\mathcal{C}^*$-algebra $\mathcal{C}$, and let $u \in A_+$ be strictly positive. Put $v = u^{\frac{1}{2}} \in \mathcal{C}_+$. The following conditions are equivalent:

(a) $(A, u)$ is an order unit $\mathcal{C}^*$-Segal algebra;

(b) there exists a self-adjoint $\mathcal{C}$-subbimodule $D$ of $\mathcal{M}(\mathcal{C})$ containing $\mathcal{C}$ and $1$ such that $A = vDv$, $vC = Cv$ and $\|vdv\| = \|d\|_{\mathcal{C}}$ for all $d \in D_h$.

In this case, the surrounding $\mathcal{C}^*$-algebra $\mathcal{C}$ is $\sigma$-unital (i.e., it contains a countable contractive approximate identity) and $E_A = vCv = uC = Cu$ and $\mathcal{M}_{\mathcal{C}}(A) = v\mathcal{M}(\mathcal{C})v = u\mathcal{M}(\mathcal{C}) = \mathcal{M}(\mathcal{C})u$. 

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$\mathcal{C}^*$-Segal algebras with order unit
An application

Let $B$ be a $C^*$-algebra and let $u \in Z(M(B))_+$ be such that $uB$ is faithful in $B$.

Put $A = uB$ and $C = A \| \cdot \|_B$. Then $A$ is a self-adjoint $C^*$-Segal algebra in $C$ under the norm $\|ux\|_u := \|x\|_B$ for $x \in B$.

It follows that $M_C(A)$ is an order unit $C^*$-Segal algebra containing $A$ isometrically as a faithful ideal (an order unitisation of $A$).

Moreover, $E_A = AC = uBC = uC$ and hence $uB$ has an approximate identity if and only if it is dense in $B$. 