Dual Banach algebras: an overview

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Dual Banach algebras: the definition

Definition

A dual Banach algebra is a pair $(\mathfrak{A}, \mathfrak{A}_*)$ of Banach spaces such that:

1. $\mathfrak{A} = (\mathfrak{A}_*)^*$;
2. $\mathfrak{A}$ is a Banach algebra, and multiplication in $\mathfrak{A}$ is separately $\sigma(\mathfrak{A}, \mathfrak{A}_*)$ continuous.

Dual Banach algebras: some examples

Examples

1. Every W^*-algebra;
2. $(M(G), \mathcal{C}_0(G))$ for every locally compact group G;
3. $(M(S), \mathcal{C}(S))$ for every compact, semitopological semigroup S;
4. $(B(G), C^*(G))$ for every locally compact group G;
5. $(\mathcal{B}(E), E \otimes^\gamma E^*)$ for every reflexive Banach space E;
6. Let $\mathfrak{A}$ be a Banach algebra and let $\mathfrak{A}^{**}$ be equipped with either Arens product. Then $(\mathfrak{A}^{**}, \mathfrak{A}^*)$ is a dual Banach algebra if and only if $\mathfrak{A}$ is Arens regular;
7. If $(\mathfrak{A}, \mathfrak{A}_*)$ is a dual Banach algebra and $\mathfrak{B}$ is a $\sigma(\mathfrak{A}, \mathfrak{A}_*)$ closed subalgebra of $\mathfrak{A}$, then $(\mathfrak{B}, \mathfrak{A}_*/{}^\perp\mathfrak{B})$ is a dual Banach algebra.

Uniqueness of the predual, I

Question

Given a dual Banach algebra $(\mathfrak{A}, \mathfrak{A}_*)$, is $\mathfrak{A}_*$ unique, i.e., if E_1 and E_2 are Banach spaces such that $(\mathfrak{A}, E_1)$ and $(\mathfrak{A}, E_2)$ are dual Banach algebras, do $\sigma(\mathfrak{A}, E_1)$ and $\sigma(\mathfrak{A}, E_2)$ coincide on $\mathfrak{A}$?

Theorem (S. Sakai, 1956)

The predual space of a W^*-algebra is unique.

Theorem (M. Daws, H. L. Pham, S. White, 2009)

Let $\mathfrak{A}$ be an Arens regular Banach algebra such that $\mathfrak{A}^{**}$ is unital. Then $\mathfrak{A}^*$ is the unique predual of $\mathfrak{A}$.

Uniqueness of the predual, II

But. . .

Example

Let $\mathfrak{A} = \ell^1$ with trivial multiplication, i.e., $fg = 0$ for all $f, g \in \ell^1$. Then (ℓ^1, c_0) and (ℓ^1, c) are both dual Banach algebras. If $\sigma(\ell^1, c_0)$ and $\sigma(\ell^1, c)$ coincided on ℓ^1, then the induced images of c_0 and c in ℓ^∞ would have to coincide. This is impossible because the unit ball of c has extreme points whereas the one of c_0 has none.

Question

Are there “more natural” examples of dual Banach algebras with non-unique preduals?

Uniqueness of the predual, III

Indeed. . .

Theorem (M. Daws, et al., 2012)

There is a family $(E_t)_{t\in\mathbb{R}}$ *of Banach spaces such that:*

1. $(\ell^1(\mathbb{Z}), E_t)$ *is a dual Banach algebra for each* $t \in \mathbb{R}$ *where* $\ell^1(\mathbb{Z})$ *is equipped with the convolution product;*
2. $E_t \cong c_0$ *for each* $t \in \mathbb{R}$*;*
3. $\sigma(\ell^1(\mathbb{Z}), E_t) \neq \sigma(\ell^1(\mathbb{Z}), E_s)$ *for* $t \neq s$*.*

Still, . . .

Even though the predual $\mathfrak{A}_*$ of a dual Banach algebra $(\mathfrak{A}, \mathfrak{A}_*)$ need not be unique, there is in many cases a canonical choice for $\mathfrak{A}_*$, e.g., $M(G)_* = \mathcal{C}_0(G)$. In such cases, we usually suppose tacitly that we are dealing with that particular $\mathfrak{A}_*$, and simply call $\mathfrak{A}$ a dual Banach algebra.

Daws' representation theorem

Recall. . .

$(\mathcal{B}(E), E \otimes^\gamma E^*)$ is a dual Banach algebra for reflexive E, as is each of its weak* closed subalgebras.

Theorem (M. Daws, 2007)

Let $(\mathfrak{A}, \mathfrak{A}_*)$ be a dual Banach algebra. Then there are a reflexive Banach space E and an isometric, $\sigma(\mathfrak{A}, \mathfrak{A}_*)$-weak* continuous algebra homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(E)$.

In short. . .

Every dual Banach algebra "is" a weak* closed subalgebra of $\mathcal{B}(E)$ for some reflexive E.

A bicommutant theorem

Question

Does von Neumann's bicommutant theorem extend to general dual Banach algebras?

Example

Let

$$\mathfrak{A} := \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}.$$

Then $\mathfrak{A} \subset \mathcal{B}(\mathbb{C}^2)$ is a dual Banach algebra, but $\mathfrak{A}'' = \mathcal{B}(\mathbb{C}^2)$.

Theorem (M. Daws, 2010)

Let $\mathfrak{A}$ be a unital dual Banach algebra. Then there are a reflexive Banach space E and a unital, isometric, weak*-weak* continuous algebra homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(E)$ such that $\pi(\mathfrak{A}) = \pi(\mathfrak{A})''$.

Banach $\mathfrak{A}$-bimodules and derivations

Definition

Let $\mathfrak{A}$ be a Banach algebra, and let E be a Banach $\mathfrak{A}$-bimodule. A bounded linear map $D : \mathfrak{A} \to E$ is called a derivation if

$$D(ab) := a \cdot Db + (Da) \cdot b \qquad (a, b \in \mathfrak{A}).$$

If there is $x \in E$ such that

$$Da = a \cdot x - x \cdot a \qquad (a \in \mathfrak{A}),$$

we call D an inner derivation.

Amenable Banach algebras

Remark

If E is a Banach $\mathfrak{A}$-bimodule, then so is E^*:

$$\langle x, a \cdot \phi \rangle := \langle x \cdot a, \phi \rangle \qquad (a \in \mathfrak{A}, \phi \in E^*, x \in E)$$

and

$$\langle x, \phi \cdot a \rangle := \langle a \cdot x, \phi \rangle \qquad (a \in \mathfrak{A}, \phi \in E^*, x \in E).$$

We call E^* a dual Banach $\mathfrak{A}$-bimodule.

Definition (B. E. Johnson, 1972)

$\mathfrak{A}$ is called amenable if, for every Banach $\mathfrak{A}$-bimodule E, every derivation $D : \mathfrak{A} \to E^*$, is inner.

Amenability for groups and Banach algebras

Theorem (B. E. Johnson, 1972)

The following are equivalent for a locally compact group G:

1. *$L^1(G)$ is an amenable Banach algebra;*
2. *the group G is amenable.*

Theorem (H. G. Dales, F. Ghahramani, & A. Ya. Helemskiĭ, 2002)

The following are equivalent:

1. *$M(G)$ is amenable;*
2. *G is amenable and discrete.*

Amenable C^*-algebras

Theorem (A. Connes, U. Haagerup, et al.)

The following are equivalent for a C^*-algebra $\mathfrak{A}$:

1. $\mathfrak{A}$ *is nuclear;*
2. $\mathfrak{A}$ *is amenable.*

Theorem (S. Wasserman, 1976)

The following are equivalent for a von Neumann algebra $\mathfrak{M}$:

1. $\mathfrak{M}$ *is nuclear;*
2. $\mathfrak{M}$ *is subhomogeneous, i.e.,*

$$\mathfrak{M} \cong M_{n_1}(\mathfrak{M}_1) \oplus \cdots \oplus M_{n_k}(\mathfrak{M}_k)$$

with $n_1, \ldots, n_k \in \mathbb{N}$ and $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$ abelian.

Virtual digaonals

Definition (B. E. Johnson, 1972)

An element $\mathbf{D} \in (\mathfrak{A} \hat{\otimes}^\gamma \mathfrak{A})^{**}$ is called a virtual diagonal for $\mathfrak{A}$ if

$$a \cdot \mathbf{D} = \mathbf{D} \cdot a \qquad (a \in \mathfrak{A})$$

and

$$a\Delta^{**}\mathbf{D} = a \qquad (a \in \mathfrak{A}),$$

where $\Delta : \mathfrak{A} \hat{\otimes}^\gamma \mathfrak{A} \to \mathfrak{A}$ denotes multiplication.

Theorem (B. E. Johnson, 1972)

$\mathfrak{A}$ is amenable if and only if $\mathfrak{A}$ has a virtual diagonal.

Normality

Definition (R. Kadison, BEJ, & J. Ringrose, 1972)

Let $\mathfrak{M}$ be a von Neumann algebra, and let E be a dual Banach $\mathfrak{M}$-bimodule. Then:

1. E is called normal if the module actions

$$\mathfrak{M} \times E \to E, \quad (a, x) \mapsto \begin{cases} a \cdot x \\ x \cdot a \end{cases}$$

   are separately weak*-weak* continuous;

2. if E is normal, we call a derivation $D : \mathfrak{M} \to E$ normal if it is weak*-weak* continuous.

Connes-amenability for von Neumann algebras

Theorem (R. Kadison, BEJ, & J. Ringrose, 1972)

Suppose that $\mathfrak{M}$ is a von Neumann algebra containing a weak* dense amenable C^*-subalgebra. Then, for every normal Banach $\mathfrak{M}$-bimodule E, every normal derivation $D : \mathfrak{M} \to E$ is inner.

Definition (A. Connes, 1976; A. Ya. Helemskiĭ, 1991)

A von Neumann algebra $\mathfrak{M}$ is Connes-amenable if, for every normal Banach $\mathfrak{M}$-bimodule E, every normal derivation $D : \mathfrak{M} \to E$ is inner.

Corollary

If $\mathfrak{A}$ is an amenable C^*-algebra, then $\mathfrak{A}^{**}$ is Connes-amenable.

Injectivity, semidiscreteness, and hyperfiniteness

Definition

A von Neumann algebra $\mathfrak{M} \subset \mathcal{B}(\mathfrak{H})$ is called

1. injective if there is a norm one projection $\mathcal{E} : \mathcal{B}(\mathfrak{H}) \to \mathfrak{M}'$ (this property is independent of the representation of $\mathfrak{M}$ on $\mathfrak{H}$);
2. semidiscrete if there is a net $(S_\lambda)_\lambda$ of unital, weak*-weak* continuous, completely positive finite rank maps such that
$$S_\lambda a \overset{\text{weak}^*}{\longrightarrow} a \qquad (a \in \mathfrak{M});$$
3. hyperfinite if there is a directed family $(\mathfrak{M}_\lambda)_\lambda$ of finite-dimensional *-subalgebras of $\mathfrak{M}$ such that $\bigcup_\lambda \mathfrak{M}_\lambda$ is weak* dense in $\mathfrak{M}$.

Connes-amenability, and injectivity, etc.

Theorem (A. Connes, et al.)

The following are equivalent:

1. $\mathfrak{M}$ *is Connes-amenable;*
2. $\mathfrak{M}$ *is injective;*
3. $\mathfrak{M}$ *is semidiscrete;*
4. $\mathfrak{M}$ *is hyperfinite.*

Corollary

A C^**-algebra* $\mathfrak{A}$ *is amenable if and only if* $\mathfrak{A}^{**}$ *is Connes-amenable.*

Amenability and Connes-amenability, I

The notions of normality and Connes-amenability make sense for every dual Banach algebra. . .

Proposition

Let $\mathfrak{A}$ be a dual Banach algebra, and let $\mathfrak{B}$ be a norm closed, amenable subalgebra of $\mathfrak{A}$ that is weak* dense in $\mathfrak{A}$. Then $\mathfrak{A}$ is Connes-amenable.

Question

Suppose that $\mathfrak{A}$ is a Connes-amenable, dual Banach algebra. Does it have a norm closed, weak* dense, amenable subalgebra?

Amenability and Connes-amenability, II

Corollary

If $\mathfrak{A}$ is amenable and Arens regular. Then $\mathfrak{A}^{**}$ is Connes-amenable.

Question

Suppose that $\mathfrak{A}$ is Arens regular such that $\mathfrak{A}^{**}$ is Connes-amenable. Is then $\mathfrak{A}$ amenable?

Theorem (VR, 2001)

Suppose that $\mathfrak{A}$ is Arens regular and an ideal in $\mathfrak{A}^{**}$. Then the following are equivalent:

1. *$\mathfrak{A}$ is amenable;*
2. *$\mathfrak{A}^{**}$ is Connes-amenable.*

Amenability and Connes-amenability, III

Corollary

Let E be reflexive and have the approximation property. Then the following are equivalent:

1. $\mathcal{K}(E)$ *is amenable;*
2. $\mathcal{B}(E)$ *is Connes-amenable.*

Example (N. Grønbæk, BEJ, & G. A. Willis, 1994)

Let $p, q \in (1, \infty) \setminus \{2\}$ such that $p \neq q$. Then $\mathcal{K}(\ell^p \oplus \ell^q)$ is not amenable. Hence, $\mathcal{B}(\ell^p \oplus \ell^q)$ is not Connes-amenable.

Normal, virtual diagonals, I

Notation

For a dual Banach algebra $\mathfrak{A}$, let $\mathcal{B}^2_\sigma(\mathfrak{A}, \mathbb{C})$ denote the separately weak* continuous bilinear functionals on $\mathfrak{A}$.

Observations

1. $\mathcal{B}^2_\sigma(\mathfrak{A}, \mathbb{C})$ is a closed submodule of $(\mathfrak{A} \otimes^\gamma \mathfrak{A})^*$.
2. $\Delta^* \mathfrak{A}_* \subset \mathcal{B}^2_\sigma(\mathfrak{A}, \mathbb{C})$, so that $\Delta^{**} : (\mathfrak{A} \otimes^\gamma \mathfrak{A})^{**} \to \mathfrak{A}^{**}$ drops to a bimodule homomorphism $\Delta_\sigma : \mathcal{B}^2_\sigma(\mathfrak{A}, \mathbb{C})^* \to \mathfrak{A}$.

Normal, virtual diagonals, II

Definition (E. G. Effros, 1988 (for von Neumann algebras))

Let $\mathfrak{A}$ be a dual Banach algebra. Then $\mathbf{D} \in \mathcal{B}^2_\sigma(\mathfrak{A}, \mathbb{C})^*$ is called a normal, virtual diagonal for $\mathfrak{A}$ if

$$a \cdot \mathbf{D} = \mathbf{D} \cdot a \qquad (a \in \mathfrak{A})$$

and

$$a \Delta_\sigma \mathbf{D} = a \qquad (a \in \mathfrak{A}).$$

Proposition

Suppose that $\mathfrak{A}$ has a normal, virtual diagonal. Then $\mathfrak{A}$ is Connes-amenable.

Normal, virtual diagonals and Connes-amenability

Question

Is the converse true?

Theorem (E. G. Effros, 1988)

A von Neumann algebra $\mathfrak{M}$ is Connes-amenable if and only if $\mathfrak{M}$ has a normal virtual diagonal.

Theorem (VR, 2003)

The following are equivalent for a locally compact group G:

1. *G is amenable;*
2. *$M(G)$ is Connes-amenable;*
3. *$M(G)$ has a normal virtual diagonal.*

Weakly almost periodic functions

Definition

A bounded continuous function $f : G \to \mathbb{C}$ is called weakly almost periodic if $\{L_x f : x \in G\}$ is relatively weakly compact in $\mathcal{C}_b(G)$. We set

$$\mathcal{WAP}(G) := \{f \in \mathcal{C}_b(G) : f \text{ is weakly almost periodic}\}.$$

Remark

$\mathcal{WAP}(G)$ is a commutative C^*-algebra. Its character space $G_{\mathcal{WAP}}$ is a compact, semitopological semigroup containing G as a dense subsemigroup. This turns $\mathcal{WAP}(G)^* \cong M(G_{\mathcal{WAP}})$ into a dual Banach algebra.

Connes-amenability without a normal, virtual diagonal

Proposition

The following are equivalent:

1. *G is amenable;*
2. *$\mathcal{WAP}(G)^*$ is Connes-amenable.*

Theorem (VR, 2006 & 2013)

Suppose that G has small invariant neighborhoods, e.g, is compact, discrete, or abelian. Then the following are equivalent:

1. *$\mathcal{WAP}(G)^*$ has a normal virtual diagonal;*
2. *G is compact.*

Minimally weakly almost periodic groups, I

Definition

A bounded continuous function $f : G \to \mathbb{C}$ is called almost periodic if $\{L_x f : x \in G\}$ is relatively compact in $\mathcal{C}_b(G)$. We set

$$\mathcal{AP}(G) := \{f \in \mathcal{C}_b(G) : f \text{ is almost periodic}\}.$$

We call G minimally weakly almost periodic (m.w.a.p.) if

$$\mathcal{WAP}(G) = \mathcal{AP}(G) + \mathcal{C}_0(G).$$

Minimally weakly almost periodic groups, II

Proposition

Suppose that G is amenable and m.w.a.p. Then $\mathcal{WAP}(G)$ has a normal virtual diagonal.

Examples

1. All compact groups are m.w.a.p.
2. $\mathrm{SL}(2, \mathbb{R})$ is m.w.a.p., but not amenable.
3. The motion group $\mathbb{R}^N \rtimes \mathrm{SO}(N)$ is m.w.a.p. for $N \geq 2$ and amenable.

Question

Does $\mathcal{WAP}(G)^*$ have a normal virtual diagonal if and only if G is amenable and m.w.a.p.?

Quasi-expectations

Theorem (J. Tomiyama, 1970)

Let $\mathfrak{A}$ be a C^*-algebra, let $\mathfrak{B}$ be a C^*-subalgebra of $\mathfrak{A}$, and let $\mathcal{E} : \mathfrak{A} \to \mathfrak{B}$ be a norm one projection, an expectation. Then

$$\mathcal{E}(abc) = a(\mathcal{E}b)c \qquad (a, c \in \mathfrak{B},\ b \in \mathfrak{A}).$$

Definition

Let $\mathfrak{A}$ be a Banach algebra, and let $\mathfrak{B}$ be a closed subalgebra. A bounded projection $\mathcal{Q} : \mathfrak{A} \to \mathfrak{B}$ is called a quasi-expectation if

$$\mathcal{Q}(abc) = a(\mathcal{Q}b)c \qquad (a, c \in \mathfrak{B},\ b \in \mathfrak{A}).$$

Quasi-expectations and injectivity

Theorem (J. W. Bunce & W. L. Paschke, 1978)

The following are equivalent for a von Neumann algebra $\mathfrak{M} \subset \mathcal{B}(\mathfrak{H})$:

1. $\mathfrak{M}$ *is injective;*
2. *there is a quasi-expectation* $\mathcal{Q} \colon \mathcal{B}(\mathfrak{H}) \to \mathfrak{M}'$*.*

"Definition"

We call a dual Banach algebra $\mathfrak{A}$ "injective" if there are a reflexive Banach space E, an isometric, weak*-weak* continuous algebra homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(E)$, and a quasi-expectation $\mathcal{Q} : \mathcal{B}(E) \to \pi(\mathfrak{A})'$.

Connes-amenability and injectivity, I

Easy. . .

Connes-amenability implies "injectivity", but. . .

Example

For $p, q \in (1, \infty) \setminus \{2\}$ with $p \neq q$, $\mathcal{B}(\ell^p \oplus \ell^q)$ is not Connes-amenable, but trivially "injective".

Definition (M. Daws, 2007)

A dual Banach algebra $\mathfrak{A}$ is called injective if, for each reflexive Banach space E and for each weak*-weak* continuous algebra homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(E)$, there is a quasi-expectation $\mathcal{Q} : \mathcal{B}(E) \to \pi(\mathfrak{A})'$.

Connes-amenability and injectivity, II

Theorem (M. Daws, 2007)

The following are equivalent for a dual Banach algebra $\mathfrak{A}$:

1. *$\mathfrak{A}$ is injective;*
2. *$\mathfrak{A}$ is Connes-amenable.*

Question

Let $\mathfrak{A}$ be a Connes-amenable, dual Banach algebra, let E be a reflexive Banach space, and let $\pi : \mathfrak{A} \to \mathcal{B}(E)$ be an isometric, weak*-weak* continuous algebra homomorphism. Is there a quasi-expectation $\mathcal{Q} \colon \mathcal{B}(E) \to \pi(\mathfrak{A})$?

And now something completely (not quite) different. . .

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